Instructions:

1. You have exactly four hours to answer questions in this examination.
2. There are 8 problems of which you must answer 6.
3. Only your first 6 problems will be graded.
4. Write only on one side of the paper, and start each question on a new page.
5. Clearly label each part of each question with the question number and the part, e.g., 1(a).
6. Write your number on every page.
7. Do not write your name anywhere on your exam.
8. You must show your work to receive credit.
9. While the eight questions are equally weighted, within a given question, the parts may have different weights.
1. Let \( X_1, \ldots, X_n \) be iid with common pdf (or pf) \( f(x) = \exp[\theta x - \psi(\theta)]h(x) \), where \( \theta \in \Theta \), some open interval in the real line.

   (a) Find the MLE \( \hat{\theta}_n \) of \( \theta \).
   (b) Find \( E_\theta(X_1) \) and \( V_\theta(X_1) \).
   (c) Find the asymptotic distribution of some suitably normalized \( \bar{X}_n \), where \( \bar{X}_n = n^{-1}\sum_{i=1}^n X_i \).
   (d) Using the delta method, find the asymptotic distribution of some suitably normalized \( \bar{\theta}_n \).
   (e) Find the UMP test for \( H_0 : \theta \leq \theta_0 \) against the alternatives \( H_1 : \theta > \theta_0 \).

2. (a) Suppose \( X \sim \text{Bin}(1, p) \), where \( p \in [0, 1] \). Assume squared error loss.
   (i) Show that \( \frac{1}{2}X + \frac{1}{4} \) is a Bayes estimator of \( p \) with constant risk \( \frac{1}{16} \).
   (ii) Show that \( \frac{1}{2}X + \frac{1}{4} \) is an admissible estimator of \( p \).
   (b) Let \( X_{11}, X_{12}, X_{21}, X_{22} \) have a joint multinomial \( (1; \theta/3, (1-\theta)/3, (1-\theta)/3, (1+\theta)/3) \) distribution, where \( \theta \in [0, 1] \). Note that exactly one of the \( X_{ij} \) is 1, and the rest are zeroes. Assume squared error loss.
   (i) Show that conditional on \( X_{11} + X_{21} = 1, \frac{3}{4}X_{11} + \frac{1}{4}X_{21} \) is an admissible estimator of \( \theta \).
   (ii) Show that conditional on \( X_{12} + X_{22} = 1, \frac{1}{2}(X_{22} - X_{12}) \) is an admissible estimator of \( \theta \).
   (iii) Show that \( T = \frac{3}{4}X_{11} + \frac{1}{4}X_{21} + \frac{1}{8}(X_{22} - X_{12}) \) is an admissible estimator of \( \theta \) conditional on either \( X_{11} + X_{21} = 1 \) or \( X_{12} + X_{22} = 1 \).
   (iv) Show that \( T \) has constant risk \( 3/16 \).
   (v) Show that the estimator \( S = \frac{3}{4}X_{11} - \frac{1}{8}(X_{12} + X_{21}) + \frac{1}{2}X_{22} \) of \( \theta \) has risk \( \frac{3}{32}(1 + \theta) \), and hence, dominates \( T \) as an estimator of \( \theta \).

3. Let \( \{Y_n, n \geq 1\} \) be a sequence of independent, nonnegative random variables and suppose that \( \sum_{n=1}^\infty Y_n \) converges almost certainly. Let \( \{X_n, n \geq 1\} \) be a sequence of random variables with \( X_n \) and \( Y_n \) having the same distribution for each \( n \geq 1 \).

   (a) Prove that \( \sum_{n=1}^\infty X_n \) converges almost certainly irrespective of the joint distributions of the \( \{X_n, n \geq 1\} \).
   (b) Demonstrate, by providing an example of sequences of random variables \( \{Y_n, n \geq 1\} \) and \( \{X_n, n \geq 1\} \), that part (a) can fail if the assumption that the \( \{Y_n, n \geq 1\} \) are nonnegative is dispensed with.

4. Let \( \{X_n, n \geq 1\} \) be a sequence of independent, mean 0, \( L^2 \) random variables and let \( S_n = \sum_{j=1}^n X_j \) and \( S_n^2 = \sum_{j=1}^n EX_j^2, n \geq 1 \). Prove that if \( s_n^2 \to \infty \), \( EX_n^2 = o(s_n^2) \), and \( \frac{S_n}{s_n} \xrightarrow{d} \mathcal{N}(0,1) \), then

\[
\frac{\max_{1 \leq j \leq n} |X_j|}{s_n} \overset{p}{\to} 0.
\]
5. Suppose that \( u \sim N(\mu, I_k) \) with \( \mu \neq 0 \). Let \( W = u'u \). It is known that \( W \) is distributed as \( \chi^2_k(\lambda) \), where \( \lambda = \frac{1}{2}u'\mu \), and \( \chi^2_k(\lambda) \) denotes the noncentral chi-squared distribution with \( k \) degrees of freedom and a noncentrality parameter \( \lambda \).

(a) Show that \( W \) can be written as

\[
W = W_1 + W_2
\]

where \( W_1 \) and \( W_2 \) are independently distributed such that \( W_1 \sim \chi^2_1(\lambda) \), \( W_2 \sim \chi^2_{k-1} \).

(b) Let \( Y_1 = e^{W_1} \). Give an approximate expression for the variance of \( Y_1 \) using the Delta Method.

6. Consider the one-way model,

\[
y_{ij} = \mu + \alpha_i + \epsilon_{ij},
\]

\( i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i \), where \( \alpha_i \) is fixed and the \( \epsilon_{ij} \) are independently distributed as normal variates with zero means and variances \( \sigma^2_i \) \( (i = 1, 2, \ldots, k) \).

It is assumed that the \( \sigma^2_i \) are known, but are not equal. Let \( \mu_i = \mu + \alpha_i \) be the mean of the \( i \)th treatment, and let \( \psi_c = \sum_{i=1}^{k} c_i \mu_i \) be a linear function of the \( \mu_i \), where \( c_1, c_2, \ldots, c_k \) are constants.

(a) Show that the simultaneous \((1 - \alpha)100\%\) confidence intervals on all linear functions of the \( \mu_i \) of the form \( \psi_c \) are given by

\[
\hat{\psi}_c \pm \sqrt{\frac{\chi^2_{\alpha, k} \sum_{i=1}^{k} \frac{c_i^2 \sigma^2_i}{n_i}}{n_i}},
\]

where \( \hat{\psi}_c = \sum_{i=1}^{k} c_i \bar{y}_i \), and \( \bar{y}_i \) is the sample mean of treatment \( i \) \( (i = 1, 2, \ldots, k) \), and \( \chi^2_{\alpha, k} \) is the upper \( \alpha \)-quantile of the chi-squared distribution with \( k \) degrees of freedom.

(b) Suppose now that the \( \sigma^2_i \) are unknown. Show how you can obtain an exact \((1 - \alpha)100\%\) confidence region on the vector \((\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_k)\).

7. Recall that distributions in an exponential dispersion family have densities of the form

\[
f(y; \theta, \phi) = \exp \left\{ \frac{1}{\phi} [\theta y - b(\theta)] + c(y; \phi) \right\},
\]

for \( \theta \in \Theta \subset \mathbb{R} \) and \( 1/\phi \in \Lambda \subset \mathbb{R} \). In a generalized linear model (GLM), we assume that \( Y_1, \ldots, Y_n \) are independent and that \( Y_i \) has density \( f(y_i; \theta_i, \phi/w_i) \), where \( f \) has the form \((\ast)\) and \( w_1, \ldots, w_n \) are known weights. This defines the random, or stochastic part of the GLM.

(a) Describe the systematic part of the GLM relating the mean \( \mu_i = E(Y_i) \) to the vector \( x_i \) of predictor values for the \( i \)th observation.

(b) Derive the maximum likelihood estimating equations (score equations) for the vector of regression coefficients \( \beta \).

(c) Derive the form of the elements of the "observed" and "expected" Fisher information matrices for \( \beta \). Express the expected Fisher information matrix in matrix form.

(d) What is the canonical link function for the density given in \((\ast)\)? Show that if \( \phi \) is known and the canonical link is used, then there is a simple sufficient statistic for the vector of regression parameters, \( \beta \), and give its form. Show also that the score equations and the observed Fisher information for \( \beta \) can be simplified when the canonical link is used and give the simplified forms.
8. The standard Cauchy (or Cauchy(0,1)) distribution has density \( f(u) = \frac{1}{\pi(1+u^2)} \), \(-\infty < u < \infty\), cumulative distribution function \( F(u) = \frac{1}{2} + \frac{1}{\pi} \arctan(u) \), and characteristic function \( \varphi(t) = e^{-|t|} \). If \( U \) has a standard Cauchy distribution, then the distribution of \( \sigma U + \mu \), where \(-\infty < \mu < \infty\), and \( \sigma > 0 \), is known as a Cauchy distribution with location parameter (median) \( \mu \) and scale parameter \( \sigma \). We will denote this distribution by the notation \( \text{Cauchy}(\mu, \sigma) \).

(a) If \( U_1 \) and \( U_2 \) are independent random variables, \( U_i \sim \text{Cauchy}(\mu_i, \sigma_i), \) \( i = 1, 2 \), and \( a_1 \) and \( a_2 \) are real numbers, what is the distribution of \( a_1 U_1 + a_2 U_2 \)? Justify (prove) your answer.

(b) Suppose that \( Y \) is a binary response following a generalized linear mixed model (GLMM) for \( E(Y|U) \), with link function \( g = F^{-1} \) (where \( F \) is the cdf of the Cauchy(0,1) distribution as given above) and linear predictor \( \eta = x^T \beta + z^T U \), where \( x \in \mathbb{R}^p \) and \( z \in \mathbb{R}^q \) are known covariates, \( \beta \in \mathbb{R}^p \) is a vector of regression parameters, and \( U = (U_1, \ldots, U_q) \) is a vector of independent random variables with \( U_i \sim \text{Cauchy}(0, \sigma_i), \) \( i = 1, \ldots, q \). Show that the marginal model for \( Y \) is a GLM, identify its link function, and express its coefficient vector in terms of the elements of the GLMM. In this context, what general statement can you make about the effect of a change in \( x_i \) on the marginal mean \( E(Y) \) versus its effect on the conditional mean \( E(Y|U) \)?