8.47 Let x_1, x_2, x_3, x_4 be the counts of the four cells

$$lik(\theta) = \left(\frac{2+\theta}{4}\right)^{x_1} \left(\frac{1-\theta}{4}\right)^{x_2+x_3} \left(\frac{\theta}{4}\right)^{x_4} \qquad n = x_1+x_2+x_3+x_4$$

$$l(\theta) = \ln lik(\theta) = n \cdot \ln\frac{1}{4} + x_1 \ln(2+\theta) + (x_2+x_3)\ln(1-\theta) + x_4 \ln\theta$$

$$\frac{\partial l}{\partial \theta} = \frac{x_1}{2+\theta} - \frac{x_2+x_3}{1-\theta} + \frac{x_4}{\theta}$$

set $\frac{\partial l}{\partial \theta} = 0 \Rightarrow \frac{x_1(1-\theta)\theta - (x_2+x_3)(2+\theta)\theta + x_4(2+\theta)(1-\theta)}{(2+\theta)(1-\theta)\theta} = 0$

$$\Rightarrow (x_1+x_2+x_3+x_4)\theta^2 - \theta(x_1 - 2x_2 - 2x_3 - x_4) - 2x_4 = 0$$

MLE of θ is the positive root of this equation

So $\hat{\theta}_{MLE} = 0.0357$

Asymptotic variance of $\hat{\theta}_{MLE} = \frac{1}{nI(\theta)}$

$$nI(\theta) = -E\frac{\partial^2 l(\theta)}{\partial^2\theta} = -E\left[-\frac{x_1}{(2+\theta)^2} - \frac{x_2+x_3}{(1-\theta)^2} - \frac{x_4}{\theta^2}\right]$$

$$= \frac{1}{(2+\theta)^2}Ex_1 + \frac{1}{(1-\theta)^2}E(x_2+x_3) + \frac{1}{\theta^2}Ex_4$$

$$= \frac{n}{(2+\theta)^2}\frac{2+\theta}{4} + \frac{n}{(1-\theta)^2}\frac{2(1-\theta)}{4} + \frac{n}{\theta^2}\cdot\frac{\theta}{4}$$

$$= \frac{n}{4}\left[\frac{1}{2+\theta} + \frac{2}{1-\theta} + \frac{1}{\theta}\right] = \frac{n(1+2\theta)}{2(2+\theta)(1-\theta)\theta}$$

$$Var(\hat{\theta}_{MLE}) = \frac{2(2+\theta)(1-\theta)\theta}{n(1+2\theta)} \quad ; \quad S^2(\hat{\theta}_{MLE}) = \frac{2(2+0.0357)(1-0.0357)\times 0.0357}{3839\times(1+2\times 0.0357)}$$

$$= 0.00003$$

b) 95% CI is $\hat{\theta}_{MLE} \pm 1.96\times\sqrt{0.00003} = (0.0247,\ 0.0467)$

8.50

(a): $n = x_1 + x_2 + x_3$; x_1, x_2, x_3 are the counts of the three cells

$$Lik(\theta) = (1-\theta)^{2x_1} [2\theta(1-\theta)]^{x_2} \theta^{2x_3}$$
$$= 2^{x_2} \theta^{x_1 + 2x_3} (1-\theta)^{2x_1 + x_2}$$

$$l(\theta) = x_2 \cdot \ln 2 + (x_2 + 2x_3) \ln\theta + (2x_1 + x_2) \ln(1-\theta)$$

$$\frac{\partial l}{\partial \theta} = \frac{x_2 + 2x_3}{\theta} - \frac{2x_1 + x_2}{1-\theta}$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow (x_2 + 2x_3)(1-\theta) - (2x_1 + x_2)\theta = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{x_2 + 2x_3}{2n} = \frac{68 + 2\times 112}{2(10+68+112)} = \frac{292}{380} \approx 0.768$$

(b) Asymptotic variance of $\hat{\theta}_{MLE} = \dfrac{1}{n I(\theta)}$

$$nI(\theta) = -E\frac{\partial^2 l}{\partial^2\theta} = \frac{1}{\theta^2} E(x_2 + 2x_3) + \frac{1}{(1-\theta)^2} E(2x_1 + x_2)$$
$$= \frac{n}{\theta^2}\left[2\theta(1-\theta) + 2\theta^2\right] + \frac{n}{(1-\theta)^2}\left[2(1-\theta)^2 + 2\theta(1-\theta)\right]$$
$$= \frac{2n}{\theta} + \frac{2n}{1-\theta} = \frac{2n}{\theta(1-\theta)}$$

Asymptotic Variance is $\dfrac{\theta(1-\theta)}{2n}$

estimated Asymptotic is $\dfrac{0.232 \times 0.768}{380} \approx 0.0005$

(C) 95% CI is $\hat{\theta}_{MLE} \pm 1.96 \times \sqrt{0.0005} = (0.746,\ 0.7897)$

99% CI is $\hat{\theta}_{MLE} \pm 2.57 \times \sqrt{0.0005}$

8.45

(a) $f(x)=\begin{cases}\frac{1}{\theta} & 0\le x\le\theta \\ 0 & \text{other}\end{cases}$ $\quad E_\theta X=\int_0^\theta \frac{x}{\theta}dx=\frac{1}{\theta}\cdot\frac{\theta^2}{2}=\frac{\theta}{2}$

$\hat{\theta}=\frac{2}{n}\sum_{i=1}^n x_i=2\bar{x}$; $E\hat{\theta}=E2\bar{X}=2EX=\theta$;

$\mathrm{Var}\,\hat{\theta}=\mathrm{Var}\,2\bar{x}=\frac{4\mathrm{Var}X}{n}=4\frac{EX^2-(EX)^2}{n}=4\frac{\frac{\theta^2}{3}-\frac{\theta^2}{4}}{n}=\frac{\theta^2}{3n}$

(b) $lik(\theta)=\prod_{i=1}^n\frac{1}{\theta}I_{\{0\le x_i\le\theta\}}=\frac{1}{\theta^n}\times I_{\{0\le\max_{1\le i\le n}x_i\le\theta\}}$

when $\theta<\max x_i$; $lik(\theta)=0$

when $\theta\ge\max x_i$; $lik(\theta)=\frac{1}{\theta^n}$ and $\frac{1}{\theta^n}$ is a decreasing function of θ, so MLE of θ is $\max_{1\le i\le n}x_i$

(c) let $\tilde{\theta}=\max_{1\le i\le n}x_i$

$F(x)=P(\tilde{\theta}\le x)=P(\max_{1\le i\le n}x_i\le x)=\prod_{i=1}^n P(x_i\le x)=(\frac{x}{\theta})^n I_{\{0\le x\le\theta\}}$

$f(x)=\frac{dF(x)}{dx}=\begin{cases}\frac{n}{\theta^n}x^{n-1} & 0\le x\le\theta \\ 0 & o/w.\end{cases}$

$E\tilde{\theta}=\int_0^\theta x\cdot\frac{n}{\theta^n}x^{n-1}dx=\frac{n}{n+1}\theta$

$\mathrm{Var}\,\tilde{\theta}=E\tilde{\theta}^2-(E\tilde{\theta})^2=\int_0^\theta\frac{n}{\theta^n}x^{n+1}dx-(\frac{n\theta}{n+1})^2=\frac{n\theta^2}{n+2}-(\frac{n\theta}{n+1})^2$

$=\frac{n\theta^2}{(n+2)(n+1)^2}$

Bias: $E\tilde{\theta}-\theta=\frac{n\theta}{n+1}-\theta=-\frac{\theta}{n+1}$

MSE : $\mathrm{Var}\,\tilde{\theta}+(E\tilde{\theta}-\theta)^2=\frac{n\theta^2}{(n+2)(n+1)^2}+\frac{\theta^2}{(n+1)^2}=\frac{2\theta^2}{(n+2)(n+1)}$

The method of moment estimator is $2\bar{x}=\hat{\theta}$

Bias: $E2\bar{x}-\theta=0$ $\quad MSE=\mathrm{Var}\,2\bar{x}+(Bias)^2=\mathrm{Var}\,2\bar{x}=\frac{\theta^2}{3n}$

$\mathrm{Var}(\tilde{\theta})=\frac{n\theta^2}{(n+2)(n+1)^2}<\mathrm{Var}\,\hat{\theta}$; $MSE(\tilde{\theta})\le MSE(\hat{\theta})$

(d) let $\theta^*=\frac{n+1}{n}\hat{\theta}_{MLE}$ $\quad E\theta^*=\frac{n+1}{n}\cdot\frac{n}{n+1}\theta=\theta$