Chapter 4

51. X and Y are independent

$$Var(XY) = E(XY)^2 - (EXY)^2 = EX^2Y^2 - (EX\cdot EY)^2$$
$$= EX^2\cdot EY^2 - (EX)^2\cdot(EY)^2 = [Var(X) + (EX)^2]\cdot[Var(Y) + (EY)^2]$$
$$- (EX)^2\cdot(EY)^2 = Var(X)\cdot Var(Y) + Var(X)\cdot(EY)^2 + Var(Y)\cdot(EX)^2$$
$$+ (EX)^2(EY)^2 - (EX)^2(EY)^2 = Var(X)\cdot Var(Y) + Var(X)(EY)^2 + Var(Y)\cdot(EX)^2$$

53. $COV(X, Y) = EXY - EX\cdot EY$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi^2} & 0 \le x^2+y^2 \le 1 \\ 0 & x^2+y^2 > 1 \end{cases}$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi^2}\, dy = \frac{2}{\pi^2}\sqrt{1-x^2} \quad -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi^2}\, dx = \frac{2}{\pi^2}\sqrt{1-y^2} \quad -1 \le y \le 1$$

$$EX = \int_{-1}^{1} \frac{2x}{\pi^2}\sqrt{1-x^2}\, dx = 0 \quad \left(\text{Because } \frac{2x}{\pi^2}\sqrt{1-x^2} \text{ is a odd function}\right)$$
$$EY = \int_{-1}^{1} \frac{2y}{\pi^2}\sqrt{1-y^2}\, dy = 0$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{1}{2\sqrt{1-x^2}} \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\ ;\ -1 < x < 1$$

$$E(Y|X) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\cdot\frac{1}{2\sqrt{1-x^2}}\, dy = 0$$

$$EXY = E[E(XY|X)] = E[X\cdot E(Y|X)] = E[X\cdot 0] = 0$$
$$COV(X, Y) = 0 - 0\cdot 0 = 0$$

But $f_Y(y) \ne f_{Y|X}(y|x) \Rightarrow X$ and Y are not independent