12.2

$$S_p^2 = \frac{SSW}{I(J-1)}$$

when $I=2$ $\quad S_p^2 = \dfrac{\sum_{j=1}^{J}(Y_{1j}-\bar{Y}_{1\cdot})^2 + \sum_{j=1}^{J}(Y_{2j}-\bar{Y}_{2\cdot})^2}{2\cdot(J-1)}$

$$= \frac{1}{2}\left[\frac{\sum_{j=1}^{J}(Y_{1j}-\bar{Y}_{1\cdot})^2}{J-1} + \frac{\sum_{j=1}^{J}(Y_{2j}-\bar{Y}_{2\cdot})^2}{J-1}\right] = \frac{1}{2}\left[S_X^2 + S_Y^2\right]$$

12.3 $I=2$; $F = \dfrac{SSB/1}{SSW/2(J-1)} = \dfrac{\sum_{i=1}^{2}\sum_{j=1}^{J}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2}{\sum_{i=1}^{2}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_{i\cdot})^2 \big/ 2(J-1)} = \dfrac{2J\sum_{i=1}^{2}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2}{\frac{1}{J-1}\sum_{j=1}^{J}(Y_{1j}-\bar{Y}_{1\cdot})^2 + \frac{1}{J-1}\sum_{j=1}^{J}(Y_{2j}-}$

$$= \frac{2J\sum_{i=1}^{2}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2}{S_X^2+S_Y^2} = \frac{2J\sum_{i=1}^{2}(\bar{Y}_{i\cdot}^2+\bar{Y}_{\cdot\cdot}^2-2\bar{Y}_{i\cdot}\bar{Y}_{\cdot\cdot})}{S_X^2+S_Y^2}$$

$$= \frac{2J\left[\sum_{i=1}^{2}\bar{Y}_{i\cdot}^2 + 2\bar{Y}_{\cdot\cdot}^2 - 2\bar{Y}_{\cdot\cdot}\sum_{i=1}^{2}\bar{Y}_{i\cdot}\right]}{S_X^2+S_Y^2} = \frac{2J\left[\sum_{i=1}^{2}\bar{Y}_{i\cdot}^2 - 2\bar{Y}_{\cdot\cdot}^2\right]}{S_X^2+S_Y^2}$$

$$= \frac{2J\left[\bar{Y}_{1\cdot}^2 + \bar{Y}_{2\cdot}^2 - 2\cdot\left(\frac{\bar{Y}_{1\cdot}+\bar{Y}_{2\cdot}}{2}\right)^2\right]}{S_X^2+S_Y^2} = \frac{2J\left[\bar{Y}_{1\cdot}^2 + \bar{Y}_{2\cdot}^2 - \frac{\bar{Y}_{1\cdot}^2}{2} - \frac{\bar{Y}_{2\cdot}^2}{2} - \bar{Y}_{1\cdot}\bar{Y}_{2\cdot}\right]}{S_X^2+S_Y^2}$$

$$= \frac{\bar{Y}_{1\cdot}^2 + \bar{Y}_{2\cdot}^2 - 2\bar{Y}_{1\cdot}\bar{Y}_{2\cdot}}{(S_X^2+S_Y^2)/J} = \frac{(\bar{Y}_{1\cdot}-\bar{Y}_{2\cdot})^2}{\frac{(S_X^2+S_Y^2)}{2}\cdot\frac{2}{J}} = \frac{(\bar{X}-\bar{Y})^2}{\frac{S_X^2+S_Y^2}{2}\cdot\frac{2}{J}} = t^2$$

So $\quad F_{1,\,2(J-1)} = t^2_{2(J-1)}$

12.4 (1) Theorem A

$$E(SS_W) = \sum_{i=1}^{I} \sum_{j=1}^{J_i} E(Y_{ij} - \bar{Y}_{i\cdot})^2 = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{J_i - 1}{J_i} \sigma^2$$

$$= \sigma^2 \sum_{i=1}^{I} (J_i - 1) \quad (\text{Use lemma A } P_{446})$$

$$E(SS_B) = \sum_{i=1}^{I} \sum_{j=1}^{J_i} E(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 = \sum_{i=1}^{I} \left[J_i E(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 \right]$$

$$= \sum_{i=1}^{I} \left[J_i \left[(\mu + \alpha_i - \mu)^2 + \frac{I-1}{I} \cdot \frac{\sigma^2}{J_i} \right] \right] = \sum_{i=1}^{I} \left[J_i \alpha_i^2 + \frac{I-1}{I} \sigma^2 \right]$$

$$= (I-1)\sigma^2 + \sum_{i=1}^{I} J_i \alpha_i^2$$

(2) Theorem B

$$\frac{SS_W}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{I} \sum_{j=1}^{J_i} (Y_{ij} - \bar{Y}_{i\cdot})^2$$

$$\because \quad \frac{1}{\sigma^2} \sum_{j=1}^{J_i} (Y_{ij} - \bar{Y}_{i\cdot})^2 \sim \chi^2_{J_i - 1}$$

There are I such sums in SS_W and they are independent of each other since the observations are independent. The sum of I independent chi-square random variables that each have $J_i - 1$ df follows a chi-square distribution with $\sum_{i=1}^{I} (J_i - 1)$ df

We know $SS_{TOT} = SS_B + SS_W$

now, we prove SS_B, SS_W are independent.

SS_W is a function of the vector U, which has elements $\sum_{j=1}^{J_i} (Y_{ij} - \bar{Y}_{i\cdot})^2 \quad i = 1, \cdots, I$

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SS_B is a function of the vector V, which has elements $\overline{Y_{i\cdot}}$ ($\overline{Y_{\cdot\cdot}}$ can be obtained from the $\overline{Y_{i\cdot}}$)

Thus, it is sufficient to show that U, V are independent. First if $i \neq i'$ $\sum_{j=1}^{J_i}(Y_{ij}-\overline{Y_{i\cdot}})^2$ and $\overline{Y_{i'\cdot}}$ are independent. Second $\sum_{j=1}^{J_i}(Y_{ij}-\overline{Y_{i\cdot}})^2$ and $\overline{Y_{i\cdot}}$ are independent. So, U, V are independent then SS_B, SS_W are independent.

We know $SS_{TOT} = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(Y_{ij}-\overline{Y_{\cdot\cdot}})^2$

$$\frac{SS_{TOT}}{\sigma^2} \sim \chi^2_{\left(\sum_{i=1}^{I} J_i\right)-1}$$

$$\because \quad \frac{SS_{TOT}}{\sigma^2} = \frac{SS_B}{\sigma^2} + \frac{SSW}{\sigma^2}$$

and $\frac{SS_B}{\sigma^2}$, $\frac{SSW}{\sigma^2}$ are independent

and $SS_{TOT}/\sigma^2 \sim \chi^2_{\left(\sum_{i=1}^{I} J_i\right)-1}$; $SSW/\sigma^2 \sim \chi^2_{\sum_{i=1}^{I}(J_i-1)}$

$$\therefore \quad SS_B/\sigma^2 \sim \chi^2_{I-1}$$

12.6 $1-P(\bigcap_{i=1}^{n} A_i) = P(\overline{\bigcap_{i=1}^{n} A_i}) = P(\bigcup_{i=1}^{n} A_i^c) \le \sum_{i=1}^{n} P(A_i^c)$

$$\therefore \quad P(\bigcap_{i=1}^{n} A_i) \ge 1-\sum_{i=1}^{n} P(A_i^c)$$

In the context of simultaneous confidence interval we have n C.I A_i is the ith C.I; A_i^c is the complement on the ith C.I.

12.7 $SS_B/\sigma^2 = \frac{J}{\sigma^2}\sum_{i=1}^{I}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2$ when $\alpha_i = 0$

$\Rightarrow \bar{Y}_{i\cdot} \overset{iid}{\sim} N(\mu, \frac{\sigma^2}{J}) \quad \therefore \frac{\sum_{i=1}^{I}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2}{\sigma^2/J} \sim \chi^2_{I-1}$

12.10 (1) ratio: $\frac{2q_{10,60}(\alpha)\frac{S_p}{\sqrt{7}}}{2t_{60}(\alpha/90)S_p\cdot\sqrt{\frac{2}{7}}} = \frac{q_{10,60}(\alpha)}{\sqrt{2}\cdot t_{60}(\alpha/90)}$

The lengthes of C.I for Tukey's and Bonferroni are larger than the length of C.I based on t

12.14 $$Y_{ijk}-\bar{Y}_{...}=(Y_{ijk}-\bar{Y}_{ij.})+(\bar{Y}_{i..}-\bar{Y}_{...})+(\bar{Y}_{.j.}-\bar{Y}_{...})$$
$$+(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}-\bar{Y}_{...})$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\bar{Y}_{...})^2=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\Big\{(Y_{ijk}-\bar{Y}_{ij.})^2+(\bar{Y}_{i..}-\bar{Y}_{...})^2+(\bar{Y}_{.j.}-\bar{Y}_{...})^2$$
$$+(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})^2+2(Y_{ijk}-\bar{Y}_{ij.})(\bar{Y}_{i..}-\bar{Y}_{...})+2(Y_{ijk}-\bar{Y}_{ij.})(\bar{Y}_{.j.}-\bar{Y}_{...}$$
$$+\cdots+2(\bar{Y}_{.j.}-\bar{Y}_{...})(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})$$

We need verify the cross products are zeroe

For example
$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}2(Y_{ijk}-\bar{Y}_{ij.})(\bar{Y}_{i..}-\bar{Y}_{...})$$
$$=\sum_{i=1}^{I}\sum_{j=1}^{J}\Big[2(\bar{Y}_{i..}-\bar{Y}_{...})\sum_{k=1}^{K}(Y_{ijk}-\bar{Y}_{ij.})\Big]=\sum_{i=1}^{I}\sum_{j=1}^{J}2(\bar{Y}_{i..}-\bar{Y}_{...})\cdot 0\Big]=0$$

Use these conditions we can show the cross product are zeroes
$$\sum_{i=1}^{I}(\bar{Y}_{i..}-\bar{Y}_{...})=0\ ;\ \sum_{j=1}^{J}(\bar{Y}_{.j.}-\bar{Y}_{...})=0\ ;\ \sum_{k=1}^{K}(Y_{ijk}-\bar{Y}_{ij.})=0$$
$$\sum_{j=1}^{J}(\bar{Y}_{ij.}-\bar{Y}_{i..})=0\ ;\ \sum_{i=1}^{I}(\bar{Y}_{ij.}-\bar{Y}_{.j.})=0.$$