1 Problem 3.4.1

(a)
As given in example 3.4.7, Π_Ω is spanned by $(1, \cdots, 1)'$ and $(t_1, \cdots, t_n)'$. Dimension of Π_Ω is $s = 2$. So if $c_1 = (1/\sqrt{n}, \cdots, 1/\sqrt{n})'$, then by Gram-Schmidt method, a vector in Π_Ω that is orthogonal to c_1 is given by

$$d_2 = (t_1, \cdots, t_n)' - [(t_1, \cdots, t_n)c_1]c_1 = (t_1 - \bar{t}, \cdots, t_n - \bar{t})'$$

$$||d_2|| = \sum_{i=1}^{n}(t_i - \bar{t})^2 \Longrightarrow c_2 = \frac{1}{\sqrt{\sum_{i=1}^{n}(t_i - \bar{t})^2}}(t_1 - \bar{t}, \cdots, t_n - \bar{t})'$$

(b)
$s = 3$, Π_Ω is spanned by $(1, \cdots, 1)'$, $(t_1, \cdots, t_n)'$ and $(t_1^2, \cdots, t_n^2)'$. With the simplifying assumptions and from (a), $c_1 = (1/\sqrt{n}, \cdots, 1/\sqrt{n})'$, $c_2 = (t_1, \cdots, t_n)'$. By Gram-Schmidt method.

$$d_3 = (t_1^2, \cdots, t_n^2)' - [(t_1^2, \cdots, t_n^2)c_1]c_1 - [(t_1^2, \cdots, t_n^2)c_2]c_2 = (t_1^2 - t_1\sum_{i=1}^{n} t_i^3 - \frac{1}{n}, \cdots, t_n^2 - t_n\sum_{i=1}^{n} t_i^3 - \frac{1}{n})'$$

So, finally $c_3 = \frac{1}{||d_3||}(t_1^2 - t_1\sum_{i=1}^{n} t_i^3 - \frac{1}{n}, \cdots, t_n^2 - t_n\sum_{i=1}^{n} t_i^3 - \frac{1}{n})'$

2 Problem 3.4.13

$\xi_{...} = \mu$, $\xi_{i..} = \mu + \alpha_i$, $\xi_{.j.} = \mu + \beta_j$, $\xi_{..k} = \mu + \gamma_k \Longrightarrow \mu = \xi_{...}$, $\alpha_i = \xi_{i..} - \xi_{...}$, $\beta_j = \xi_{.j.} - \xi_{...}$, $\gamma_k = \xi_{..k} - \xi_{...}$. Note that X_{ijk} is the LSE of ξ_{ijk}. Then by theorem 3.4.4, $\hat{\mu} = X_{...}$, $\hat{\alpha}_i = X_{i..} - X_{...}$,$\hat{\beta}_j = X_{.j.} - X_{...}$, $\hat{\gamma}_k = X_{..k} - X_{...}$ are UMVU estimators.
Viewed as a special case of (3.4.4), $s = I + J + K - 3 = I + J + K - 2$.

3 Problem 3.4.16

(a)
In order to obtain the LSE of θ, we minimize

$$Q(\theta) = ||x - \xi||^2 = ||x - \theta A||^2 = xx^T - x(\theta A)^T - (\theta A)x^T + (\theta A)(\theta A)^T$$

$$\Rightarrow \nabla Q(\theta) = -xA^T - Ax^T + 2AA^T\theta^T = -2Ax^T + 2AA^T\theta^T =^{set} 0$$

$$\Rightarrow xA^T = \hat{\theta}AA^T \Leftrightarrow \hat{\theta} = xA^T(AA^T)^{-1}.$$

Thus, $\hat{\theta} = xA^T(AA^T)^{-1}$ is the LSE of θ.
(b)
From Theorem 4.8, $\hat{\theta}$ is a function of $\hat{\xi}$, the least square estimate of ξ. From the proof of Theorem

4.4, we see that $\hat{\xi}$ is a function of the complete sufficient statistics. Hence, θ is a function of the complete sufficient statistics, so it is UMVUE for its expectation. Note that E$(\hat{\theta}) = \theta$. Thus, $\hat{\theta}$ is the UMVUE of θ.

4 Problem 4.1.9

(a)
Let $X_1, \ldots, X_n$ be iid Poisson distribution $P(\lambda)$, and let λ has Gamma distribution $\Gamma(g, \alpha)$. Note that the expectation and variance of a RV $\lambda \sim \Gamma(g, \alpha)$ are

$$E(\lambda) = g\alpha \qquad\qquad Var(\lambda) = g\alpha^2.$$

Moreover, also notice that $\bar{X} \sim P(\lambda)$ is the UMVUE for λ with $E(\bar{X}) = Var(\bar{X}) = \lambda$. According to Example 4.1.3, the posterior distribution is $\pi(\lambda|\bar{x}) \sim \Gamma(g + n\bar{x}, \frac{\alpha}{1+n\alpha})$. Thus, we obtain the Bayes estimator under squared error loss to be

$$\delta_{\alpha,g} = E(\lambda|\bar{x}) = (g + n\bar{x})\frac{\alpha}{1+n\alpha} = \frac{g\alpha}{1+n\alpha} + n\bar{x}\frac{\alpha}{1+n\alpha}$$

which is a weighted average of $g\alpha$, the estimator of λ before any observations are taken, and $\bar{X}$, the estimator without consideration of a prior.

(b)
(i)$n \to \infty$, E$(\delta_{\alpha,g}) \to \lambda$, Var$(\delta_{\alpha,g}) \to 0 \Rightarrow \delta_{\alpha,g} \to \lambda$ in probability.
(ii)$\alpha \to \infty$ and $g \to 0$, then $\delta_{\alpha,g} \to \bar{x}$.
Now, if both (i) and (ii) holds then E$(\delta_{\alpha,g}) \to \lambda$, Var$(\delta_{\alpha,g}) \to 0 \Rightarrow \delta_{\alpha,g} \to \lambda$ in probability.

5 Problem 4.1.11

$\pi(\lambda) = \frac{1}{\lambda} I(\lambda > 0)$, $f(\underline{x}|\lambda) = \prod_{i=1}^n \left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$. Then the posterior is $\pi(\lambda|\underline{x}) \propto e^{-n\lambda}\lambda^{\sum x_i - 1}$, which is kernel of $Gamma(\sum x_i, 1/n)$. So our condition for the posterior to be proper is that $\sum_{i=1}^n x_i > 0$.

6 Problem 4.2.3

From $(4.2.4)$, we know $E_\theta \delta_\Lambda(x) = \frac{nb^2\theta + \sigma^2\mu}{nb^2+\sigma^2}$, $Var_\theta \delta_\Lambda(x) = \frac{nb^4\sigma^2}{n^2b^4+2nb^2\sigma^2+\sigma^4}$

(a)
As $n \longrightarrow \infty$, $E_\theta \delta_\Lambda(x) \longrightarrow \theta$ and $Var_\theta \delta_\Lambda(x) \longrightarrow 0$. So $\delta_\Lambda(x) \longrightarrow \theta$ in probability as $n \longrightarrow \infty$.

(b)
As $b \longrightarrow 0$, $E_\theta \delta_\Lambda(x) \longrightarrow \mu$ and $Var_\theta \delta_\Lambda(x) \longrightarrow 0$. So $\delta_\Lambda(x) \longrightarrow \mu$ in probability as $b \longrightarrow 0$.

(c)
For fixed n, as $b \longrightarrow \infty$, $\delta_\Lambda(x) \longrightarrow \bar{X}$ in distribution.

7 Problem 4.2.16

(a)
Under squared error loss, we obtain the Bayes estimator $g(p) = p$ as

$$\delta_\Lambda(x) = E(p|x) = \frac{\int_0^1 p^{x+a}(1-p)^{n-x+b-1}dp}{\int_0^1 p^{x+a-1}(1-p)^{n-x+b-1}dp} = \frac{x+a}{a+b+n}$$

Under the improper prior $[p(1-p)]^{-1} I_{(0,1)}(p)$ (which implies $a = b = 0$).
To get the generalized Bayes estimator, we minimize

$$\text{E}\{(p-\delta(x))^2|x\} = \binom{n}{x} \int_0^1 p^{x-1}(1-p)^{n-x-1}dp$$

For $0 < x < n$ (which makes it a proper posterior), we obtain our generalized Bayes estimator to be

$$\frac{\int_0^1 p^{x}(1-p)^{n-x-1}dp}{\int_0^1 p^{x-1}(1-p)^{n-x-1}dp} = \frac{x}{n}$$

For $x = 0$ and $x = n$, $\text{E}\{(p-\delta(0))^2|x=0\}$ and $\text{E}\{(p-\delta(n))^2|x=n\}$ respectively are finite.
Thus, $\delta(x) = \frac{x}{n}$ is the generalized Bayes estimator.
(b)
For $X \sim N(\theta, 1)$ and $\pi(\theta) = 1$, our posterior density becomes $\pi(\theta|x) = f(x|\theta)$. Hence,

$$\delta(x) = E(\theta|x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta e^{-\frac{(\theta-x)^2}{2}} d\theta = x$$

Thus, $\delta(x) = x$.

8 Problem 4.3.2

(a)
Let $X_1, \ldots, X_n \sim^{iid} \Gamma(a, b)$ where a is known. Let $\eta = -\frac{1}{b}$, the density of X can be expressed as

$$f_b(x) = exp\left\{-\frac{1}{b}x - log(b^a)\right\} \frac{x^{a-1}}{\Gamma(a)} I(x \geq 0) = exp\left\{\eta x - A(\eta)\right\} h(x)$$

where $A(\eta) = a \cdot log(b)$. According to (4.3.19) the conjugate prior family is

$$\pi(\eta|k,\mu) \propto exp\{k\mu\eta - kA(\eta)\} = exp\left\{-\frac{k\mu}{b} + log(\frac{1}{b^{ka}})\right\} = \frac{1}{b^{ka}} e^{-\frac{k\mu}{b}}$$

which we recognize as the kernel of $IG(ka-1, k\mu)$ (for $ka > 1$ and $k\mu > 0$ under the support $b > 0$). Thus, the conjugate prior for $\eta = -\frac{1}{b}$ is equivalent to an inverted gamma on b.

(b)
Note: Let $b \sim InvGam(\alpha,\beta) \Rightarrow p_{\alpha,\beta}(b) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{b}\right)^{\alpha+1} e^{-\frac{\beta}{b}} I(b > 0)$ where $E(b) = \frac{\beta}{\alpha-1}$. Moreover,

$$E(\frac{1}{b}) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{1}{b^{\alpha+2}} e^{-\frac{\beta}{b}} = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta}$$

(which makes senses since $\frac{1}{b} \sim Gamma(\alpha, \frac{1}{\beta}) \Rightarrow E(\frac{1}{b}) = \frac{\alpha}{\beta}$) and

$$E(\frac{1}{b^2}) = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\beta^2}$$

Let $X_1, \ldots, X_n \sim^{iid} \Gamma(a,b)$ where $b \sim InvGam(c,d)$. Note that $\pi(b|x) \propto \frac{1}{b^{an+c+1}} e^{-\frac{1}{b}(\sum x_i + d)}$ which implies $b|x \sim InvGam(an+c, \sum x_i + d)$. First, let's consider the case where $L(b,\delta) = (b-\delta)^2$. From Corollary 4.1.2

$$\delta(x) = E(b|x) = \frac{\sum x_i + d}{an+c-1}$$

. Lastly, let's consider the case where $L(b,\delta) = \frac{1}{b^2}(b-\delta)^2$. Again, from Corollary 4.1.2

$$\delta(x) = \frac{E(\frac{1}{b}|x)}{E(\frac{1}{b^2}|x)} = \frac{\sum x_i + d}{an+c+1}$$

.

9 Problem 4.3.8

(a)
Note: from Lemma 1.5.15, Stein's indentity is given to be

$$E\left\{\left[\frac{h'(X)}{h(X)} + \sum_{i=1}^{s} \eta_i T_i'(X)\right] g(X)\right\} = -Eg'(X)$$

Let $g(X) = 1 \Rightarrow g'(X) = 0$. From Stein's indentity,

$$E_\eta\left\{\left[\frac{h'(X)}{h(X)} + \sum_{i=1}^{s} \eta_i T_i'(X)\right] g(X)\right\} = 0$$

(where $\frac{h'(X)}{h(X)} = \frac{\delta}{\delta x_j} log(h(X))$ and $T_i' = \frac{\delta}{\delta x_j} T_i(X)$)

$$\Rightarrow \sum_{i=1}^{s} \eta_i E_\eta(\frac{\delta}{\delta x_j} T_i(X)) = E_\eta(-\frac{\delta}{\delta x_j} log(h(X)))$$

(b)
Let $X_i \sim^{iid} \Gamma(a,b)$ where a is known. The density is

$$f(\mathbf{x}|b) = \frac{(\Pi x_i)^{a-1}}{(\Gamma(a))^n} exp\left\{-an \cdot log(b) - \frac{1}{b}\sum x_i\right\} = h(x) * exp\left\{-A(\eta) + \eta_i \sum T_i\right\}$$

that is, $\eta_i = \frac{1}{b}$ and $T_i = -x_i$. Notice that

$$E_\eta(-\frac{\delta}{\delta x_j} log(h(X))) = \sum_{i=1}^{s} \eta_i E_\eta(\frac{\delta}{\delta x_j} T_i(X)) = \frac{1}{b}\sum E(\frac{\delta}{\delta x_j}(-x_i)) = -\frac{1}{b}$$

which implies $\frac{\delta}{\delta x_j} log(h(X))$ is an unbiased estimator of $\frac{1}{b}$. Let's consider

$$\frac{\delta}{\delta x_j} log(h(X)) = \frac{\delta}{\delta x_j}\left(-nlog(\Gamma(a)) + (a-1)\sum log(x_i)\right) = \frac{a-1}{x_j}.$$

Thus, $\frac{a-1}{x_j}$ is an unbiased estimator of $\frac{1}{b}$ for all j. Moreover, $(a-1)\sum(x_j^{-1}/n)$ is also unbiased for $1/b$.

(c)
Let $X_i \sim^{iid} Beta(a,b)$. Let's assume b is known. Using part(a), we have

$$(a-1)E(\frac{1}{x_j}) = (b-1)E(\frac{1}{1-x_j})$$

Note that $E(\frac{1}{x_j}) = \frac{a+b-1}{a-1} \Rightarrow (a+b-1) = (b-1)E(\frac{1}{1-x_j}) \Rightarrow a = \frac{(b-1)}{n}E(\sum \frac{x_j}{1-x_j})$.
Thus, the unbiased estimator for a is $\frac{(b-1)}{n}(\sum \frac{x_j}{1-x_j})$
Similar argument is applied to the case where a is known.

10 Problem 4.3.12

(a)

$$\begin{aligned} E[(a\bar{X}+b)-\mu]^2 &= a^2E(\bar{X}-\mu)^2 + E[b+(a-1)\mu]^2 + 2a\text{E}(\bar{X}-\mu)[b+(a-1)\mu] \\ &= a^2Var(\bar{X}) + [(a-1)\mu+b]^2 \end{aligned}$$

(b)
If μ is unbounded, then $MSE(a\bar{X}+b) = a^2Var(\bar{X}) + [(a-1)\mu+b]^2$ is also unbounded if $a \neq 1$.

(c)
If μ is bounded, the Bayes estimator can have finite MSE.

11 Problem 4.4.1

(a)
$X|p \sim Bin(n,p), p \sim Beta(\alpha,\alpha)$. Then $p|X \sim Beta(\alpha+x, \alpha+n-x)$. We know $g(X) = n-X$, $\bar{g}(p) = 1-p$, $g^*(u) = 1-u$ and $\delta(X) = E(p|X) = \frac{\alpha+x}{2\alpha+n}$. So $g^*(\delta(X)) = 1 - \frac{\alpha+x}{2\alpha+n} = \frac{\alpha+n-x}{2\alpha+n} = \delta(g(X))$. So, this Bayes rule is equivariant.

(b)

$$g^*(\delta(X)) = 1 - \delta(X) = \frac{\int_0^1 p^x(1-p)^{n-x}\pi(p)dp - \int_0^1 p^{x+1}(1-p)^{n-x}\pi(p)dp}{\int_0^1 p^x(1-p)^{n-x}\pi(p)dp}$$

Let $p_1 = 1-p$ then

$$g^*(\delta(X)) = \frac{\int_0^1 (p_1)^{n-x+1}(1-p_1)^x\pi(1-p_1)dp_1}{\int_0^1 (p_1)^{n-x}(1-p_1)^x\pi(1-p_1)dp_1}$$

But $\pi(\cdot)$ is symmetric about $1/2$, so $\pi(1-p_1) = \pi(p_1)$. Then

$$g^*(\delta(X)) = \frac{\int_0^1 (p_1)^{n-x+1}(1-p_1)^x\pi(p_1)dp_1}{\int_0^1 (p_1)^{n-x}(1-p_1)^x\pi(p_1)dp_1} = \delta(g(X))$$