STA7346 - HWK 1

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Problem 1.2

Let $X_1, \ldots, X_n$ be uncorrelated random variables with common expectation θ and variance σ^2. Then, among all linear estimators $\sum \alpha_i X_i$ of θ satisfying $\sum \alpha_i = 1$, the mean $\bar{X}$ has the smallest variance.

Solution. $Var(\sum \alpha_i X_i) = \sum Var(\alpha_i X_i) = \sum \alpha_i^2 Var(X_i) = \sigma^2 \sum \alpha_i^2$, so we must minimize $\sum \alpha_i^2$. We will do this using the method of Lagrange multipliers, where we wish to minimize the function $f(\alpha_1, \ldots, \alpha_n) = \sum \alpha_i^2$ subject to the constraint $g(\alpha_1, \ldots, \alpha_n) = \sum \alpha_i - 1 = 0$.

Define

$$\Lambda = f(\alpha_1, \ldots, \alpha_n) - \lambda g(\alpha_1, \ldots, \alpha_n) = \sum \alpha_i^2 - \lambda \sum \alpha_i + \lambda.$$

Taking derivatives and setting them equal to zero, we get

$$\frac{\partial \Lambda}{\partial \alpha_j} = 2\alpha_j - \lambda = 0$$

for $j = 1, \ldots, n$. We can sum these n equations to get

$$2 \sum \alpha_j - n\lambda = 0,$$

and solving for λ yields

$$\lambda = \frac{2 \sum \alpha_j}{n} = \frac{2}{n}.$$

We can substitute this back into each of the n equations and solve for α_j to yield

$$\alpha_j = \frac{1}{n}$$

for $j = 1, \ldots, n$. It should be noted that this choice could yield either the maximum or the minimum. To check, we can compare this choice to an alternative choice of $\alpha_1 = 1,\ \alpha_2 = \cdots = \alpha_n = 0$. This alternative choice would yield

$$\sum \alpha_i^2 = 1 > \frac{1}{n} = \sum \left(\frac{1}{n}\right)^2,$$

which implies that our solution is indeed the minimum, and not the maximum. Therefore the mean $\bar{X}$ has the smallest variance among all estimators of the type specified.

Comment: When you want to use *Lagrange multipliers* to find minimum, you need to verify that your solution is indeed the minimum.

Problem 1.3

In the preceding problem, minimize the variance of $\sum \alpha_i X_i (\sum \alpha_i = 1)$

(a) When the variance of X_i *is* σ^2/α_i *(*α_i *known).*

Solution. $Var(\sum \alpha_i X_i) = \sum \alpha_i^2 Var(X_i) = \sum \alpha_i \sigma^2 = \sigma^2$. Thus any choice of positive $\alpha_1, \ldots, \alpha_n$ that meet the constraint will yield the same variance.

(b) When the X_i *have common variance* σ^2 *but are correlated with common correlation coefficient* ρ*.*

Solution.

$$\begin{aligned}
Var(\sum \alpha_i X_i) &= \sum \alpha_i^2 Var(X_i) + 2 \sum\sum_{i<j} \alpha_i \alpha_j Cov(X_i, X_j) \\
&= \sigma^2 \sum \alpha_i^2 + 2 \sum\sum_{i<j} \alpha_i \alpha_j \rho \sigma^2 \\
&= \sigma^2 \left(\sum \alpha_i^2 + 2 \sum\sum_{i<j} \alpha_i \alpha_j + (2\rho - 2) \sum\sum_{i<j} \alpha_i \alpha_j \right) \\
&= \sigma^2 \left(\left(\sum \alpha_i \right)^2 + 2(\rho - 1) \sum\sum_{i<j} \alpha_i \alpha_j \right) \\
&= \sigma^2 \left(1 + 2(\rho - 1) \sum\sum_{i<j} \alpha_i \alpha_j \right).
\end{aligned}$$

Since $\rho - 1 < 0$, minimizing the variance is equivalent to maximizing $\sum\sum_{i<j} \alpha_i \alpha_j$. We will do this using the method of Lagrange multipliers, where we wish to minimize the function $f(\alpha_1, \ldots, \alpha_n) = \sum\sum_{i<j} \alpha_i \alpha_j$ subject to the constraint $g(\alpha_1, \ldots, \alpha_n) = \sum \alpha_i - 1 = 0$.

Define

$$\Lambda = f(\alpha_1, \ldots, \alpha_n) - \lambda g(\alpha_1, \ldots, \alpha_n) = \sum\sum_{i<j} \alpha_i \alpha_j - \lambda \sum \alpha_i + \lambda.$$

Taking derivatives and setting them equal to zero, we get

$$\frac{\partial \Lambda}{\partial \alpha_j} = \sum \alpha_i - \alpha_j - \lambda = 0$$

for $j = 1, \ldots, n$. Substituting using the constraint yields

$$1 - \alpha_j - \lambda = 0$$

for $j = 1, \ldots, n$. We can sum these n equations to get

$$n - \sum \alpha_j - n\lambda = 0,$$

and solving for λ yields

$$\lambda = \frac{n - \sum \alpha_j}{n} = \frac{n-1}{n}.$$

We can substitute this back into each of the n equations and solve for α_j to yield

$$\alpha_j = \frac{1}{n}$$

for $j = 1, \ldots, n$. It should be noted that this choice could yield either the maximum or the minimum. To check, we can compare this choice to an alternative choice of $\alpha_1 = 1,\ \alpha_2 = \cdots = \alpha_n = 0$. This alternative choice would yield

$$\sum \sum_{i<j} \alpha_i \alpha_j = 0 < \frac{n-1}{2n} = \sum \sum_{i<j} \left(\frac{1}{n}\right)^2,$$

which implies that our solution is indeed the maximum, and not the minimum. (Remember that to minimize the variance of the estimator, we must maximize $\sum \sum_{i<j} \alpha_i \alpha_j$.) Therefore the mean $\bar{X}$ has the smallest variance among all estimators of the type specified.

Problem 1.12

a) Let

$$f(x) = \frac{1}{2}\frac{(k-1)}{(1+|x|)^k}, k \geq 2$$

To show that f is a probability density note that since $k \geq 2$

$$f(x) = \frac{1}{2}\frac{(k-1)}{(1+|x|)^k} \geq 0$$

Note that $f(x) = f(-x)$ for all $x \in \Re$, f is an even function. Observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \frac{1}{2}\frac{(k-1)}{(1+|x|)^k}dx \\
&= \int_{0}^{\infty} \frac{(k-1)}{(1+|x|)^k}dx \\
&= \int_{0}^{\infty} \frac{(k-1)}{(1+x)^k}dx \\
&= \int_{1}^{\infty} (k-1)u^{-k}du \qquad\qquad (u = x+1) \\
&= -u^{1-k}|_1^{\infty} \\
&= 0-(-1) \\
&= 1
\end{aligned}$$

Since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$, it follows that f is a p.d.f..

To show that all the moments of order $p < k-1$ are finite, note that it suffices to show that $E|X|^{k-2} < \infty$ because this would imply that all moments of order $p \le k-2 < k-1$ are finite. Observe that

$$
\begin{aligned}
EX^{k-2} &\le E|X|^{k-2} \\
&= \int_{-\infty}^{\infty} |x|^{k-2} f(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\frac{(k-1)|x|^{k-2}}{(1+|x|)^k}dx \\
&= \int_{0}^{\infty} \frac{(k-1)|x|^{k-2}}{(1+|x|)^k}dx \\
&= \int_{0}^{\infty} (k-1)(\frac{x}{1+x})^{k-2}(1+x)^{-2}dx \\
&< \int_{0}^{\infty} (k-1)(1+x)^{-2}dx \qquad (k \ge 2 \text{ and } \frac{x}{1+x} < 1 \Rightarrow (\frac{x}{1+x})^{k-2} < 1) \\
&= \int_{1}^{\infty} (k-1)u^{-2}du \qquad (u = x+1) \\
&= -(k-1)u^{-1}|_1^{\infty} \\
&= 0 + k - 1 \\
&< \infty
\end{aligned}
$$

Thus, the $(k-2)^{th}$ moment is finite and this implies that all moments of order $p < k-1$ are finite.

b) We need to show that $2\int_{-\infty}^{\infty} f^2(x)dx < f(0)$. Note that

$$k-1 < k - \frac{1}{2} = \frac{(2k-1)}{2} \Rightarrow \frac{k-1}{(2k-1)} < \frac{1}{2}$$

Since $k-1 > 0$, we obtain

$$\frac{(k-1)^2}{(2k-1)} < \frac{k-1}{2} = f(0)$$

Let's compute $2\int_{-\infty}^{\infty} f^2(x)dx$.

$$
\begin{aligned}
2\int_{-\infty}^{\infty} f^2(x)dx &= \int_{-\infty}^{\infty} \frac{1}{2}\frac{(k-1)^2}{(1+|x|)^2k}dx \\
&= \int_0^{\infty} \frac{(k-1)^2}{(1+x)^{2k}}dx \\
&= \int_1^{\infty} (k-1)^2 u^{-2k} du \qquad\qquad (u = x+1) \\
&= \frac{(k-1)^2 u^{1-2k}}{1-2k}|_1^{\infty} \\
&= \frac{(k-1)^2}{(2k-1)} < \frac{k-1}{2} = f(0)
\end{aligned}
$$

Thus, $2\int_{-\infty}^{\infty} f^2(x)dx < f(0)$.

Problem 1.13

(a) If X is binomial $b(p,n)$, show that

$$
E\left|\frac{x}{n} - p\right| = \binom{n-1}{k-1} p^k (1-p)^{n-k+1} \quad \textit{for} \quad \frac{k-1}{n} \le p \le \frac{k}{n}.
$$

Solution.

$$
\begin{aligned}
E\left|\frac{X}{n} - p\right| &= \sum_{x=0}^{n} \left|\frac{x}{n} - p\right| \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} + \sum_{x=k}^{n} \left(\frac{x}{n} - p\right) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} - \sum_{x=k}^{n} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} \\
&= 2\sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} - \sum_{x=0}^{n} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} \\
&= 2\sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x}
\end{aligned}
$$

because the second term is just $E\left(p - \frac{X}{n}\right) = 0$.

We now claim that

$$\sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} = \binom{n-1}{k-1} p^k (1-p)^{n-k+1}.$$

This claim will be proven by induction. First, for $k = 1$, by inspection, both sides are equal to $p(1-p)^n$. Now assume the claim to be true for some k. We must show that it is then also true for $k + 1$.

$$\begin{aligned}
\sum_{x=0}^{k} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} &= \sum_{x=0}^{k-1} \left(p - \frac{x}{n}\right) \binom{n}{x} p^x (1-p)^{n-x} \\
&\quad + \left(p - \frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{n-1}{k-1} p^k (1-p)^{n-k+1} \\
&\quad + \left(p - \frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{p^k (1-p)^{n-k}}{(n-k)!} \\
&\quad \times \left[\frac{(n-1)!(1-p)}{(k-1)!} + \frac{n!}{k!}\left(p - \frac{k}{n}\right)\right] \\
&= \frac{p^k (1-p)^{n-k}}{(n-k)!} \\
&\quad \times \left[\frac{k(n-1)!(1-p)}{k!} + \frac{(n-1)!}{k!}(np - k)\right] \\
&= \frac{p^k (1-p)^{n-k} (n-1)!}{k!(n-k)!} \left(k(1-p) + (np - k)\right) \\
&= \frac{p^{k+1} (1-p)^{n-k} (n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{p^{k+1} (1-p)^{n-k} (n-1)!}{k!(n-k-1)!} \\
&= \binom{n-1}{(k+1)-1} p^{k+1} (1-p)^{n-(k+1)+1}.
\end{aligned}$$

Thus the claim is true for $k + 1$. Therefore, by induction, the claim is true for all $k \geq 1$.

Thus we have shown that

$$E\left|\frac{x}{n}-p\right| = \binom{n-1}{k-1}p^k(1-p)^{n-k+1} \quad \text{for} \quad \frac{k-1}{n} \le p \le \frac{k}{n}.$$

Problem 2.1

Let $A_1, A_2, \cdots \in \mathcal{A}$, where $\mathcal{A}$ is a σ-algebra. Clearly, by the definition of σ-algebra, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. We also have $A_i^c \in \mathcal{A}$, since σ-algebra's are closed under complementation. This means $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{A}$, and hence, by De Morgan's Law,

$$\bigcap_{i=1}^{\infty} A_i \;=\; (\bigcup_{i=1}^{\infty} A_i^c)^c \;\in\; \mathcal{A}.$$

Problem 2.9

If f is integrable with respect to μ, so is $|f|$, and $\left|\int f d\mu\right| \le \int |f| d\mu$. [Hint: Express $|f|$ in terms of f^+ and f^-.]

Solution. $f = f^+ - f^-$. Since f is integrable with respect to μ, both f^+ and f^- are integrable with respect to μ. Therefore $|f| = f^+ + f^-$ is integrable with respect to μ as well, and

$$\left|\int f d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right| \le \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right|$$

by the triangle inequality. Then

$$\left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right| = \int f^+ d\mu + \int f^- d\mu$$

since both are positive, and

$$\int f^+ d\mu + \int f^- d\mu = \int (f^+ + f^-) d\mu = \int |f| d\mu.$$

Thus $\left|\int f d\mu\right| \le \int |f| d\mu$.

Problem 3.8

Independently choose $X \sim Exponential(\lambda), Y \sim Exponential(\mu)$, and define

$$Z = \min\{X, Y\} \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases} .$$

Let $z \geq 0$.

$$\begin{aligned}
P(Z \leq z, W = 1) &= P(X \leq z, X \leq Y) \;=\; \int_{x=0}^{z} \int_{y=x}^{\infty} \frac{1}{\lambda} e^{\frac{-1}{\lambda}x} \frac{1}{\mu} e^{\frac{-1}{\mu}y} \, dy \, dx \\
&= \frac{1}{\lambda\mu} \int_{x=0}^{z} e^{\frac{-1}{\lambda}x} \left[-\mu e^{\frac{-1}{\mu}y} \right]_{y=x}^{\infty} dx \;=\; \frac{1}{\lambda} \int_{x=0}^{z} e^{-x(\frac{1}{\lambda}+\frac{1}{\mu})} \, dx \\
&= \frac{1}{\lambda} \frac{-\lambda\mu}{\lambda+\mu} e^{-x(\frac{1}{\lambda}+\frac{1}{\mu})} \Big|_{x=0}^{z} \;=\; \frac{\mu}{\lambda+\mu} \left(1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \right) \\
P(Z \leq z, W = 0) &= P(Y \leq z, Y \leq X) \;=\; \frac{\lambda}{\lambda+\mu} \left(1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \right)
\end{aligned}$$

$$\begin{aligned}
P(Z \leq z) &= P(Z \leq z, W = 1) + P(Z \leq z, W = 0) \;=\; \frac{\lambda+\mu}{\lambda+\mu} \left(1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \right) \\
&= 1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \\
P(W = 1) &= P(X \leq Y) \;=\; \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{1}{\lambda} e^{\frac{-1}{\lambda}x} \frac{1}{\mu} e^{\frac{-1}{\mu}y} \, dy \, dx \;=\; \frac{1}{\lambda} \int_{x=0}^{\infty} e^{-x(\frac{1}{\lambda}+\frac{1}{\mu})} \, dx \\
&= \frac{-\mu}{\lambda+\mu} e^{-x(\frac{1}{\lambda}+\frac{1}{\mu})} \Big|_{x=0}^{\infty} \;=\; \frac{-\mu}{\lambda+\mu}(0-1) \;=\; \frac{\mu}{\lambda+\mu} \\
P(W = 0) &= 1 - P(W = 1) \;=\; 1 - \frac{\mu}{\lambda+\mu} \;=\; \frac{\lambda}{\lambda+\mu}
\end{aligned}$$

$$\begin{aligned}
P(Z \leq z, W = 1) &= \frac{\mu}{\lambda+\mu} \left(1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \right) \;=\; P(W = 1)\, P(Z \leq z) \\
P(Z \leq z, W = 0) &= \frac{\lambda}{\lambda+\mu} \left(1 - e^{-z(\frac{1}{\lambda}+\frac{1}{\mu})} \right) \;=\; P(W = 0)\, P(Z \leq z)
\end{aligned}$$

Thus, the coded variables Z and W are independent.

Problem 4.1

If the distributions of a positive random variable X form a scale family, show that the distributions of $\log X$ form a location family.

Solution. Let U be a random variable with a fixed distribution F_1, and let $X = bU$ with $b > 0$. Then the distributions of X form a scale family. Now consider the distributions of $\log X = \log b + \log U$. (Note that $\log X$ is a monotonically increasing function of X.)

$$P(\log X \leq y) = P(\log b + \log U \leq y) = P(\log U \leq y - \log b) = F_2(y - c),$$

where $c = \log b$ and F_2 is the distribution of $\log U$. Thus the distributions of $\log X$ form a location family.

Problem 4.3

Let $U \sim U(0,1)$. Define the element $g_a \in G$ of the transformation group where $g_a(U) = U^a$.
(1) Clearly if $a, b > 0$, then $ab > 0$. So

$$(g_a \circ g_b)(U) = (U^b)^a = U^{ab} = g_{ab}(U)$$

Clearly, g_{ab} is a member of G, so G is closed under completion.
(2) With the use of the associative property of multiplication, we see

$$\begin{aligned}((g_a \circ g_b) \circ g_c)(U) &= (g_{ab} \circ g_c)(U) = g_{ab}(g_c(U)) = (U^c)^{ab} = \\ (U^{bc})^a &= g_a(g_{bc}(U)) = (g_a \circ g_{bc})(U) = (g_a \circ (g_b \circ g_c))(U)\end{aligned}$$

(3) Clearly, g_1 is the identity element of G. It follows that for any $a > 0$,

$$\begin{aligned}(g_a \circ g_1)(U) = (U^1)^a &= (U^a)^1 = (g_1 \circ g_a)(U) \\ &= U^a = g_a(U).\end{aligned}$$

(4) For any $a > 0$, we have $\frac{1}{a} > 0$, and so $g_{a^{-1}} \in G$.

$$\begin{aligned}g_1(U) = U &= (U^a)^{\frac{1}{a}} = (g_{a^{-1}} \circ g_a)(U) \\ &= (U^{\frac{1}{a}})^a = (g_a \circ g_{a^{-1}})(U).\end{aligned}$$

Thus, $g_{a^{-1}}$ is the inverse of g_a.
Hence, G satisfies the conditions and describes a legitimate group.

Restrict attention to $u, x \in [0,1]$.

$$\begin{aligned}F_U(u) &= \begin{cases} 0, & \text{for } u < 0, \\ u, & \text{for } u \in [0,1], \\ 1, & \text{for } u > 1. \end{cases} \\ F_X(x) &= P(X \le x) = P(U^\alpha \le x) = P(U \le \sqrt[\alpha]{x}) = F_U(\sqrt[\alpha]{x}) \\ &= \begin{cases} 0, & \text{for } x < 0, \\ \sqrt[\alpha]{x}, & \text{for } x \in [0,1], \\ 1, & \text{for } x > 1. \end{cases} \\ f_X(x) &= \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{\alpha} x^{\frac{1}{\alpha}-1}, & \text{for } x \in [0,1], \\ 0, & \text{for } x \notin [0,1]. \end{cases}\end{aligned}$$

Problem 4.15

The following two families of distributions are not group families:

(a) The class of binomial distributions $b(p, n)$ with n fixed and $0 < p < 1$.

Solution. Let $U \sim b(p_0, n)$ for some p_0, where $0 < p_0 < 1$. Now consider the distributions of $g(U)$, where $g : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$. There are only finitely many such transformations g, so the family of distributions of $g(U)$ will consist of only finitely many distributions. However, the family of binomial distributions $b(p, n)$ with n fixed and $0 < p < 1$ consists of infinitely many distributions since there are infinitely many possibilities for p. Thus the family of binomial distributions is not a group family.

(b) The class of Poisson distributions Poisson(λ), $0 < \lambda$.

Solution. Let $V \sim \text{Poisson}(1)$. Suppose the class of Poisson distributions Poisson(λ), $0 < \lambda$, forms a group family. Then there must exist a transformation $h : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$, such that $h(V) \sim \text{Poisson}(2)$, and such that h^{-1} exists. Then

$$P\left(h(V) = 0\right) = \frac{1}{0!} 2^0 e^{-2} = e^{-2}.$$

Now apply h^{-1} to get

$$P\left(h(V) = 0\right) = P\left(h^{-1}\left(h(V)\right) = h^{-1}(0)\right) = P\left(V = h^{-1}(0)\right) = e^{-2}.$$

Let $y = h^{-1}(0)$. Clearly $y \in \{0, 1, 2, \ldots\}$. Then $P(V = y) = e^{-2}$, where $V \sim \text{Poisson}(1)$. This means that

$$e^{-2} = P(V = y) = \frac{1}{y!} 1^y e^{-1} = \frac{1}{y!} e^{-1},$$

which implies that $y! = e$, which cannot be satisfied by any $y \in \{0, 1, 2, \ldots\}$. Thus there is no transformation h such that $h(V) \sim \text{Poisson}(2)$. Therefore the family of Poisson distributions is not a group family.