Lecture 32 Friday April 21

# Randomized Complete Block Design Ch. 21

Lec 31 last time 1. Design, reasons for blocking, how to randomize, Risk Premium E
3 Treatments, 5 Blocks

2. Model, notation for observed means & effects etc pp. 171-17

3. Sums of squares, df, ANOVA table (incl. MS, E(MS))

4. F tests for Treatment & Block Factors pp. 176-177

Note: p. 178 has data, and row & col. avg's (skip "by hand" part)

5. Statistical analysis in R pp. 183, 184 (skip 185-189
190-192 through F test for blocks

## Model for randomized block design with one observation per cell

We assume the additive two-way model, the same as the additive model discussed for the two-way factorial design. However, different notation is used to emphasize the nature of the randomized block design, which is different from the nature of the factorial experiment.

This is a two-factor ANOVA model for the case where there is only 1 observation per cell — $n = 1$.

We <u>have</u> <u>to</u> assume additivity in order to use the interaction mean square as our error mean square.

by contrast w/ "random effects"

**Model for randomized blocks design with fixed effects for blocks**

Rows represent blocks; columns represent treatments. Let $n_b$ = no. of blocks = no. of rows in the two-way layout. Let $r$ = no. of treatments = no. of columns in the two-way layout.

Model equation for observation $Y_{ij}$, , which is the response for the $j^{\text{th}}$ treatment in the $i^{\text{th}}$ block:

row, block

column, treatment

$$Y_{ij} = \mu_{..} + \rho_i + \tau_j + \epsilon_{ij}$$

don't need index k because only 1 obs/cell

where

$\mu_{..}$ is a constant,

$\rho_i$ are constants for the block (row) effects; $\sum_{i=1}^{n_b} \rho_i = 0$

$\tau_j$ are constants for the treatment effects; $\sum_{j=1}^{r} \tau_j = 0$

Assumptions on the $\epsilon_{ij}$: independent, normally distributed, with mean 0 and constant variance $\sigma^2$

## Notation for observed means

Let $\bar{Y}_{i.} = \frac{\sum_{j=1}^{r} Y_{ij}}{r}$ be the mean of observations in the ith block, $i = 1, \ldots, n_b$

let $\bar{Y}_{.j} = \frac{\sum_{i=1}^{n_b} Y_{ij}}{n_b}$ be the mean of observations in the jth treatment, $j = 1, \ldots, r$

and let $\bar{Y}_{..} = \frac{1}{n_b r} \sum_{i=1}^{n_b} \sum_{j=1}^{r} Y_{ij}$, be the grand mean.

Further,

let $\hat{\rho}_i = \bar{Y}_{i.} - \bar{Y}_{..}$

and $\hat{\tau}_j = \bar{Y}_{.j} - \bar{Y}_{..}$

**Fitted values:**

$$\begin{aligned}\hat{Y}_{ij} &= \overline{Y}_{..} + (\overline{Y}_{i.} - \overline{Y}_{..}) + (\overline{Y}_{.j} - \overline{Y}_{..}) \\ &= \hat{\mu}_{..} + \hat{\rho}_i + \hat{\tau}_j\end{aligned}$$

Note: $\hat{Y}_{ij}$ simplifies to $\overline{Y}_{i.} + \overline{Y}_{.j} - \overline{Y}_{..}$

**Residuals:**

$$e_{ij} = Y_{ij} - \hat{Y}_{ij} = \underline{Y_{ij} - \overline{Y}_{i.} - \overline{Y}_{.j} + \overline{Y}_{..}}$$

same as the interaction effects
in two-way model of Ch.19 & Project 2

## Sums of squares

There are three sums of squares, for Blocks, Treatments, and Block × Treatment interaction. Below are the formulas for "by-hand" calculation of the sums of squares.

Two-way $n \geq 2$

like SSA $\leftarrow$ $$\text{SSBL} = \sum_{i=1}^{n_b} r(\overline{Y}_{i.} - \overline{Y}_{..})^2$$

like SSB $\leftarrow$ $$\text{SSTR} = \sum_{j=1}^{r} n_b(\overline{Y}_{.j} - \overline{Y}_{..})^2$$

like SSAB $\longleftarrow$ $$\text{SSBL.TR} = \sum_{i=1}^{n_b}\sum_{j=1}^{r}(Y_{ij} - \overline{Y}_{i.} - \overline{Y}_{.j} + \overline{Y}_{..})^2$$

IMPORTANT: With only one observation per cell, the interaction sum of squares is used as the error sum of squares.

*Degrees of freedom*:

df(Blocks) $= n_b - 1$

df(Treatment) $= r - 1$

df(Bl.Tr) $= (n_b - 1)(r - 1)$

Recall: A *Mean Square* is a sum of squares, divided by its degrees of freedom. A *mean square* is a statistic and has a sampling distribution.

*Mean Squares in RCB Design, formulas*

$$\text{MSBL} = \frac{\text{SSBL}}{n_b - 1}$$

$$\text{MSTR} = \frac{\text{SSTR}}{r - 1}$$

$$\text{MSBL.TR} = \frac{\text{SSBL.TR}}{(n_b - 1)(r - 1)}$$

*Expected Mean Squares*

$$\text{E(MSBL)} = \sigma^2 + r\frac{\sum \rho_i^2}{n_b - 1}$$

$$\text{E(MSTR)} = \sigma^2 + n_b\frac{\sum \tau_j^2}{r - 1}$$

$$\text{E(MSBL.TR)} = \sigma^2$$

**ANOVA table**

| **Source of Variation** | Sum of Squares | df | Mean Square | Expected Mean Square |
|---|---|---|---|---|
| Blocks | SSBL | $n_b - 1$ | MSBL | $\sigma^2 + r\frac{\sum \rho_i^2}{n_b-1}$ |
| Treatments | SSTR | $r - 1$ | MSTR | $\sigma^2 + n_b\frac{\sum \tau_j^2}{r-1}$ |
| Error | SSBL.TR | $(n_b - 1)(r - 1)$ | MSBL.TR | $\sigma^2$ |

# $F$ Test for Treatment Factor

*Hypotheses to be tested:*

$H_0$: $\tau_1 = \tau_2 = \cdots = \tau_r = 0$ vs.

$H_a$: At least one $\tau_i$ is not zero.

*Test Statistic*:

$$F_{\text{obs}} = \frac{\text{MSTR}}{\text{MSBL.TR}}$$

*Null distribution*: $F_{r-1,(n_b-1)(r-1)}$

*P-value*: area to the right of the observed $F$ statistic, under the curve of the null distribution

*Decision rule*: Reject $H_0$ at level $\alpha$, if $P < \alpha$. If you do reject $H_0$, go on to do followup analysis of treatment (column) means, using whichever of Bonferroni, Scheffé, or Tukey methods is appropriate.

# $F$ Test for Blocks

Since Block is a "nuisance factor," we are not too interested in testing for block effects.

However, we would like to see a significant F test for blocks, because this gives an indication that this factor was useful for blocking.

The main statistical reason for forming blocks is to **reduce error variability** by comparing every treatment within each group of units in a block.

If the F test for Block is significant, this indicates that it was valuable to "take out" block effects from the error.

*Example of randomized blocks design: Risk premium*

## Data

| | Method ($j$) | | | |
|---|---|---|---|---|
| Block — | Utility | Worry | Comparison | Average |
| 1 (oldest) | 1 | 5 | 8 | 4.7 |
| 2 | 2 | 8 | 14 | 8.0 |
| 3 | 7 | 9 | 16 | 10.7 |
| 4 | 6 | 13 | 18 | 12.3 |
| 5 (youngest) | 12 | 14 | 17 | 14.3 |
| Average | 5.6 | 9.8 | 14.6 | 10.0 |

$$\hat{Y}_{11} = \overline{Y}_{1.} + \overline{Y}_{.1} - \overline{Y}_{..} = 4.7 + 5.6 - 10.0 = .3$$

$$e_{11} = Y_{11} - \hat{Y}_{11} = 1 - .3 = .7$$

*Ex: Risk Premium* Create the dataframe, fit the randomized blocks model, and get the ANOVA table.

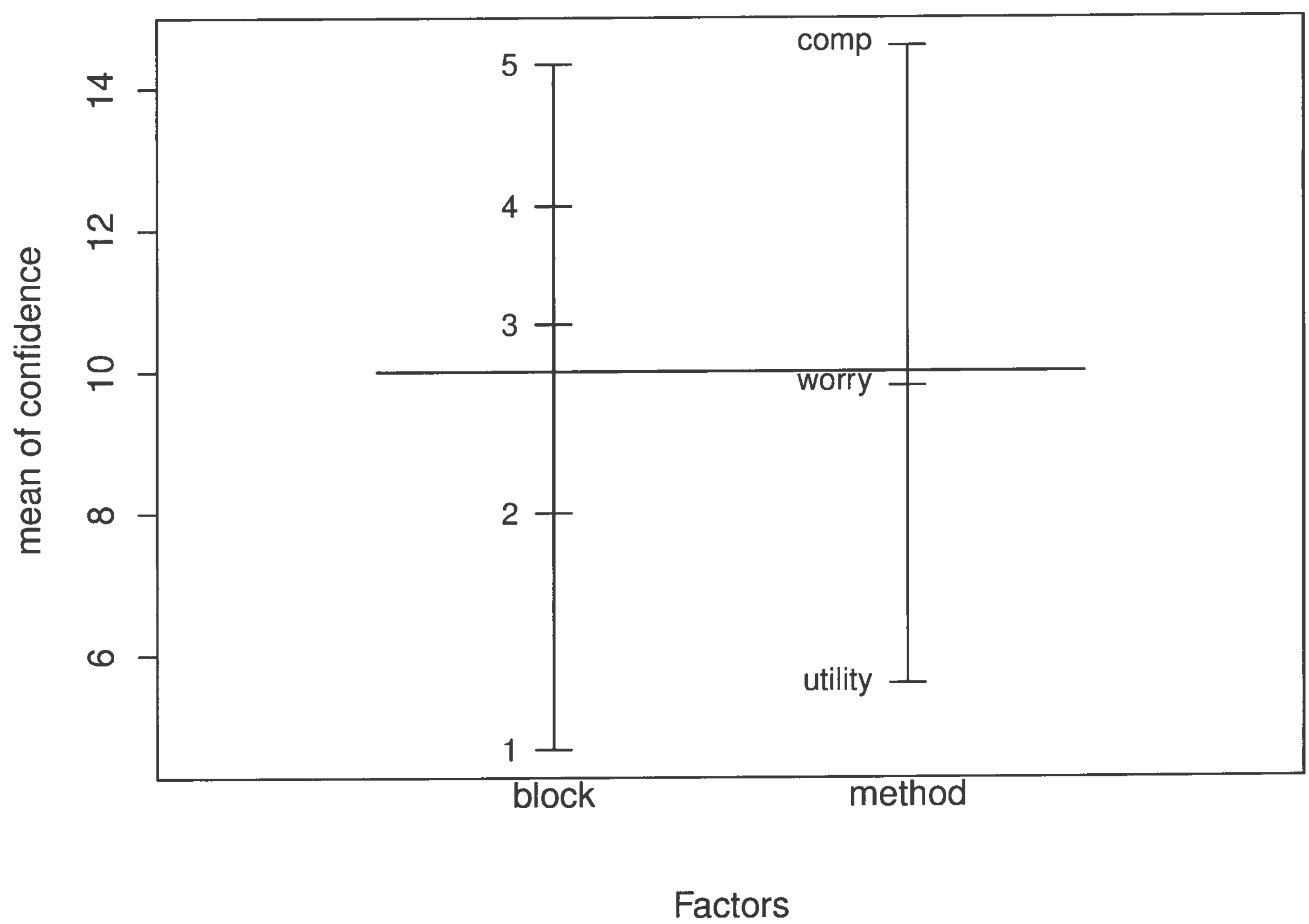
```
> confidence <- c(1,2,7,6,12,5,8,9,13,14,8,14,16,18,17)
> method <- factor(rep(c("utility","worry","comp"),c(5,5,5)))
> block <- factor(rep(1:5,3))
> riskprem <- data.frame(confidence=confidence,
+                              method=method,block=block)
> riskprem.aov <- aov(confidence ~ block + method,
+                             data=riskprem)
> anova(riskprem.aov)

          Df  Sum Sq Mean Sq F value    Pr(>F)
block      4 171.333  42.833  14.357 0.0010081 **
method     2 202.800 101.400  33.989 0.0001229 ***
Residuals  8  23.867   2.983
```

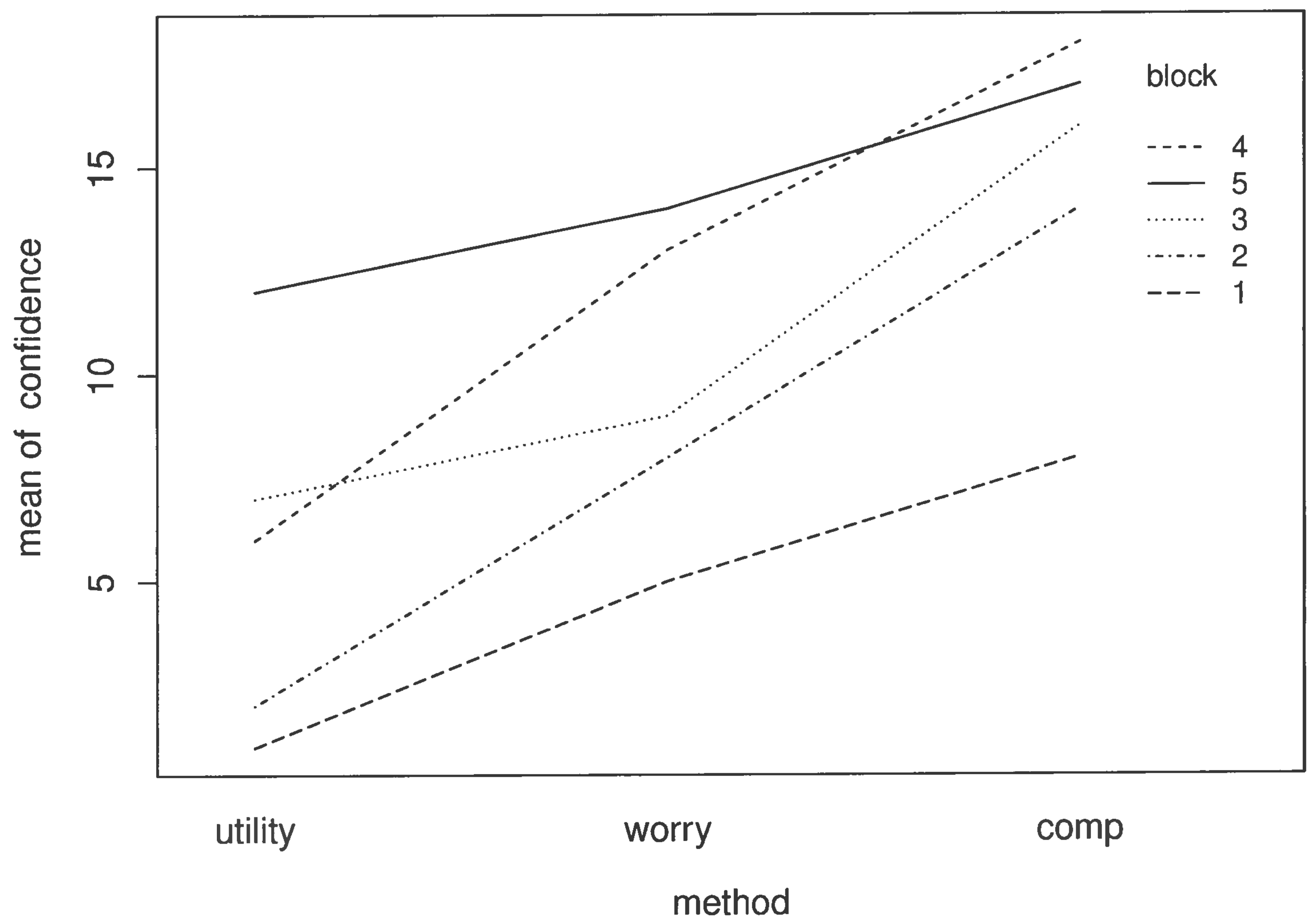
We should look at relevant plots, and check assumptions before drawing conclusions from the F tests.

Get the plots of factor level means for both blocks and treatment.

```
> plot.design(confidence ~ block + method)
```



```
> interaction.plot(method, block, response=confidence)
```



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*Ex Risk premium* Wrap up the analysis.

Based on the residual plots (not shown in the notes), the assumption of normality of errors appears reasonably well-satisfied.

From the plot of residuals versus fitted values, and from Tukey's 1 df test, we have justified use of the additive model.

So, now proceed to carry out the analysis of treatment effects, based on the model fit we have reported earlier.

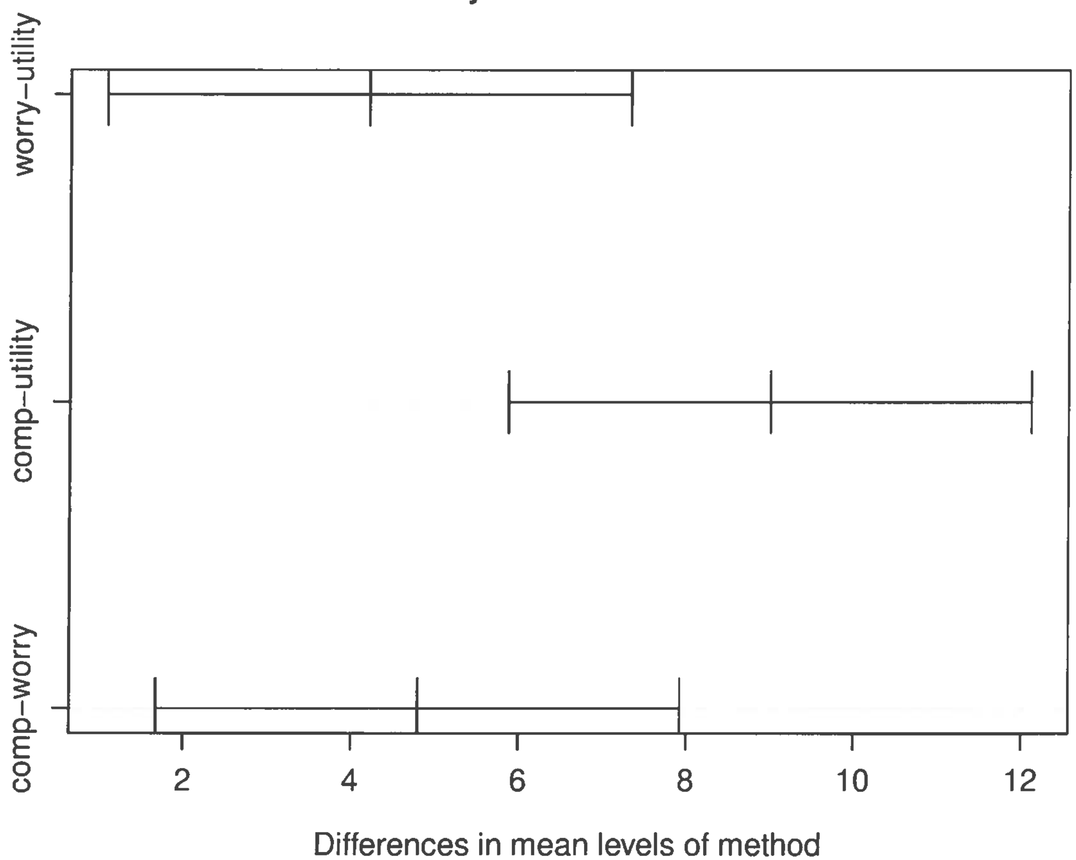
F test for
(treatment)

average

The $F$ test for differences among executives' confidence in the three methods of quantifying risk premium is highly significant ($F_{\text{obs}} = 33.99, P = .00012.$)

For followup analysis, let's do all pairwise comparisons of means by Tukey's method:

95% family-wise confidence level



Ex. Risk premium

Use Tukey's HSD method to form 95% simultaneous CIs for all pairwise differences.

Solution.

① Critical constant is

$$T = \frac{1}{\sqrt{2}}\, q\,(.95,\ r=3,\ (n_b-1)(r-1) = 4(2) = 8)$$

In R:

```
> 1/sqrt(2) * qtukey(.95, nmeans = 3, df = 8)
[1] 2.85744
```

② $$\hat{SD}(\cancel{\ }\bar{Y}_{\cdot i} - \bar{Y}_{\cdot j}) = \sqrt{\frac{2\hat{\sigma}^2}{n_b}} = \sqrt{\frac{2\,MSBL.TR}{5}}$$

$$= \sqrt{\frac{2(2.99)}{5}} = 1.0936$$

③ HSD (Allowance): $2.8574(1.0936) = 3.1248$

④ Form chart of: Parameter, Estimate, CI (Use $\bar{Y}_{\cdot j}$'s p.174)

1st line: $\mu_{\cdot 3} - \mu_{\cdot 2}$, $14.6 - 9.8 = 4.8$, $4.8 \pm 3.1$ or $(1.7, 7.9)$

*Ex Risk premium* We now look at information obtained from this experiment about the blocking variable (age). Since the F test for Blocks is highly significant ($F_{\text{obs}} = 14.357, P = .0010081$), it appears that the use of Age as block variable led to improved efficiency.

We can in fact compute a more specific estimate of *efficiency of blocking*.

**Efficiency of Blocking**

Call the error variance with randomized blocks $\sigma_b^2$. For a completely randomized design **with the same experimental units**, a different error variance would have been obtained; call this error variance $\sigma_r^2$.

The ratio $\sigma_r^2/\sigma_b^2$ is used to measure the *relative efficiency* of blocking.