Lecture 4 Wednesday January 18

- Expected value (EV) & variance (Var)
- EV, Var of linear combinations

An outcome is a single possibility eg Heads in one toss of a coin

Probability for Applied Statistics

Undefined terms: Sample space, outcomes, events ← An event is any subset of the sample space.

set of all possible outcomes of an ~~random~~ experiment involving chance
eg $\{H, T\}$ for one coin toss

A *probability distribution* p is a function on the space of all events, which satisfies three axioms.

1. Probability of the whole sample space is 1.
2. For any event A, $0 \le P(A) \le 1$
3. Addition rule for mutually exclusive events

Definition. Random variable A function from the sample space to the real line.

Eg $$X = \begin{cases} 1 & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$$

Expected Value and Variance: Discrete Case

Expected Value: If X is a discrete random variable, then the *expected value of* X, denoted $E(X)$, is the weighted average of all the possible values of X, where the weights are given by the probabilities.

In symbols, if the possible values are $x_1, x_2, x_3, \ldots$, and the probabilities are $p_1, p_2, p_3, \ldots$, then

$$E(X) = \sum x_i p_i.$$

Variance: If X is a discrete random variable then the *variance of* X, denoted $\mathrm{Var}(X)$, is the weighted average of the squared distance between the possible values of X and the expected value of X, where the weights are given by the probabilities.

In symbols, if the possible values are $x_1, x_2, x_3, \ldots$, the probabilities are $p_1, p_2, p_3, \ldots$, and the expected value is μ, then

$$\mathrm{Var}(X) = \sum \underbrace{(x_i - \mu)^2}_{\text{squared distance between } x_i \text{ and } \mu} p_i \qquad \left[\text{in other words, } \mathrm{Var}(X) = E\big((X-\mu)^2\big)\right].$$

Standard Deviation: For a random variable X with variance $\mathrm{Var}(X)$, the SD or standard deviation of X is $\sqrt{\mathrm{Var}(X)}$

Example. **Mean and SD of the distribution for the number of aces.** Let X represent the number of aces in a randomly selected deal of two cards in the game of Texas hold 'em.. Here is the probability distribution for the random variable X:

X_i (Value of X)	0	1	2
p_i (Probability)	0.8507	0.1448	0.0045

- Find $E(X) = \mu_X$, the mean of the probability distribution of X.
- Find the standard deviation of the probability distribution of X.

Answers: In R:

```
> x <- c(0,1,2)
> p <- c(.8507, .1448, .0045)
> mu <- sum(x * p); mu   # .1538
> varx <- sum((x-mu)^2 * p); varx   #
> sdx <- sqrt(varx); sdx   # .3730
```

μ_X

σ_X

Note: $(\mu_X - \sigma_X, \mu_X + \sigma_X)$ contains 0 but not 1 or 2, so contains 85% of the distribution, about what you'd expect for this right-skewed distribution. This is a rough check on the calculations.

Expected Value and Variance: Continuous Case

For continuous r.v. X, $E(X)$ and $\mathrm{Var}(X)$ are defined similarly to the discrete case, but require calculus for the actual formula.

Normal r.v., $X \sim N(\mu, \sigma)$, we have $EX = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

For normal r.v.s, there is an important interpretation of the standard deviation (see p. 14).

Rules for Expected Value and Variance of Linear Combinations

Suppose $X_1, X_2, \ldots, X_n$ are random variables with means $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively. Suppose $a_1, a_2, \ldots, a_n$ and c are constants. Then:

Whether or not the X's are independent,

$$\begin{aligned} E(a_1X_1 + \cdots + a_nX_n + c) &= a_1E(X_1) + \cdots + a_nE(X_n) + c \\ &= a_1\mu_1 + \cdots + a_n\mu_n + c \\ &= \Sigma_{i=1}^n a_i\mu_i + c \end{aligned}$$

If the X's are independent,

$$\begin{aligned} \text{Var}(a_1X_1 + \cdots + a_nX_n + c) &= a_1^2\text{Var}(X_1) + \cdots + a_n^2\text{Var}(X_n) \\ &= a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2 \\ &= \Sigma_{i=1}^n a_i^2\sigma_i^2 \end{aligned}$$

Question:

Call the true period of a pendulum μ. Suppose there are two measurement techniques (two different clocks) for measuring the period. Call the measurement for clock 1 X_1, and the mst. for clock 2 X_2. Suppose that both methods are unbiased. That is, $E(X_1) = \mu$, $E(X_2) = \mu$. However, the SDs are different. $SD(X_1) = .2$ and $SD(X_2) = .1$. Find the mean and SD of the weighted average, $.25X_1 + .75X_2$.

Answer:

$$E(.25X_1 + .75X_2) = .25\mu + .75\mu = \mu,$$

by the rule for expected value of a linear combination of r.v.'s
And, assuming the two msts. are made independently of each other,

$$\begin{aligned}\mathrm{Var}(.25X_1 + .75X_2) &= .25^2(\mathrm{Var}(X_1)) + .75^2(\mathrm{Var}(X_2)) \\ &= .0625(.04) + .5625(.01) = .008125\end{aligned}$$

So, SD$(.25X_1 + .75X_2) = \sqrt{.008125} = .09014$