

- **difference equations (2 points)** Find a closed form solution to the difference equations below given initial conditions $Z_0 = 0$ and $Z_1 = 1$.

(a) $Z_t - 1.1Z_{t-1} + .3Z_{t-2} = 0$

(b) $Z_t + 1.6Z_{t-1} + .64Z_{t-2} = 0$

- (a) This part is not graded.
 (b) The characteristic polynomial is $\alpha(z) = 1 + 1.6z + .64z^2 = .64(z + 1.25)^2$. The general solution of Z_t , for the case with equal roots, yields

$$Z_n = \left(\frac{-4}{5}\right)^n (c_1 + c_2 n)$$

From the initial conditions, we see that $c_1 = 0$ and $c_2 = -5/4$. Therefore the final solution is

$$Z_n = \left(\frac{-4}{5}\right)^{n-1} n$$

- **#3.1(a,b) (2 points)** Find the ACF and PACF and plot the ACF ρ_k for $k = 0, 1, 2, 3, 4, 5$ for each of the following models where the a_t is a Gaussian white noise process.

(a) $Z_t - .5Z_{t-1} = a_t$

(b) $Z_t + .98Z_{t-1} = a_t$

- (a) The ACF of a general AR(1) with parameter ϕ is $\rho_k = \phi^k$, and the PACF is

$$\phi_{kk} = \begin{cases} \phi, & k = 1 \\ 0, & k > 1 \end{cases}$$

Therefore the ACF is $\rho_k = .5^k$ and the PACF is $\phi_{kk} = .5$ when $k = 1$ and 0 for $k > 1$. (The plots have been omitted.)

- (b) This part is not graded.

- **#3.4 (2 points)** (a) Show that the ACF ρ_k for the AR(1) process satisfies the difference equation

$$\rho_k - \phi_1 \rho_{k-1} = 0 \quad \text{for } k \geq 1$$

- (b) Find the general expression for ρ_k .

- (a) Starting from the general equation $Z_t = \phi_1 Z_{t-1} + a_t$ and multiplying through by Z_{t-k} gives

$$Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + a_t Z_{t-h}$$

Now taking expectations throughout produces

$$\gamma(h) = \phi_1 \gamma(h-1) + 0$$

Finally, dividing by $\gamma(0)$ throughout and rearranging gives the desired equation:

$$\rho_k - \phi_1 \rho_{k-1} = 0 \quad \text{for } k \geq 1$$

- (b) This part is not graded.

○ **#3.5 (4 points)** Consider the AR(2) process $Z_t = Z_{t-1} - .25Z_{t-2} + a_t$.

- (a) Calculate ρ_1
- (b) Use ρ_0 and ρ_1 as starting values and the difference equation to obtain the general expression for ρ_k .
- (c) Calculate the values ρ_k for $k = 1, 2, \dots, 10$.

• (a) From equation (3.1.20) on p. 42 of the text, we see

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} = \frac{1}{1 + .25} = \frac{4}{5}$$

(b) The autocorrelation function satisfies the recursion equation

$$\rho_k - \rho_{k-1} + .25\rho_{k-2} = 0$$

The characteristic equation is $\alpha(z) = 1 - z + .25z^2 = .25(z - 2)^2$. The general solution of this difference equation, for the case with equal roots, yields

$$\rho_k = \left(\frac{1}{2}\right)^k (c_1 + c_2k)$$

From the initial conditions $\rho_0 = 1$ and $\rho_1 = 4/5$, we see that $c_1 = 1$ and $c_2 = 3/5$. Therefore the final solution is

$$\rho_k = \left(\frac{1}{2}\right)^k \left(1 + \frac{3}{5}k\right)$$

(c) This part is not graded.