Consider the Gibbs sampler (7.1.1):

Set $X_0 = x_0$, for $t = 1, 2, \ldots$, generate

$Y_t \sim f_{X|Y}(x_t | x_{t-1})$

$X_t \sim f_{Y|X}(y_t | y_t)$

where $f_{X|Y}$ and $f_{Y|X}$ are the conditional distributions.

a) From the description above, it is easy to see that the pair $(X_t, Y_t)$ depends only on the pair $(x_{t-1}, y_{t-1})$, so $(X_t, Y_t)$ is a Markov chain.

The transition kernels for $(X_t)$ and $(Y_t)$ are:

$$K(x, x^*) = \int f_{Y|X}(y | x)f_{X|Y}(x^* | y)dy$$

$$K(y, y^*) = \int f_{X|Y}(x | y)f_{Y|X}(y^* | x)dx$$

So, $(X_t)$ and $(Y_t)$ also depend only on $x_{t-1}$ and $y_{t-1}$ respectively, so $(X_t)$ and $(Y_t)$ are Markov chains.

b)

$$\int K(x, x^*) f_X(x)dx = \int \int f_{Y|X}(y | x)f_{X|Y}(x^* | y)dyf_X(x)dx =$$

$$= \int f_{X|Y}(x^* | y)\int f_{Y|X}(y | x)f_X(x)dx dy =$$

$$= \int f_{X|Y}(x^* | y)\int f_{XY}(x, y)dxdy = \int f_{XY}(x^* | y)f_Y(y)dy = f_X(x^*)$$

So $f_X(x)$ is the invariant density of $(X_t)$.

$$\int K(y, y^*) f_Y(y)dy = \int \int f_{X|Y}(x | y)f_{Y|X}(y^* | x)dxf_Y(y)dy =$$

$$= \int f_{Y|X}(y^* | x)\int f_{X|Y}(x | y)f_Y(y)dxdy =$$

$$= \int f_{Y|X}(y^* | x)\int f_{XY}(x, y)dxdy = \int f_{Y|X}(y^* | x)f_X(x)dx = f_Y(y^*)$$

So $f_Y(y)$ is the invariant density of $(Y_t)$.