STA6934 Problem 5.22abc Solution by Joseph Powers

(a) Be warned, I'm of the opinion that there are several typos in this problem. Clearly, \( \{X_i\} \) is an iid sequence since, for a fixed value of \( i \), \( X_i \) depends only on \( Z_i \), and \( \{Z_i\} \) is iid. The iidness of the \( Z \)'s also means that the success probability will be constant. Hence, the \( X \)'s really are Bernoulli random variables. Below, let \( \theta \) be the vector \((\zeta, \sigma^2)\). First compute the success probability:

\[
P(X_i = 1 \mid \theta) = P(Z_i > u) = P(z > \frac{u - \zeta}{\sigma}) = \Phi\left(\frac{u - \zeta}{\sigma}\right) = p \tag{1}
\]

(b) Consider \( \{z_i\} \) to be the complete date (since given the \( z \)'s the \( x \)'s are redundant). The computation of the complete data likelihood is then easy, thanks to independent normality.

\[
L_c(\theta \mid z) = \prod_{i=1}^{n} f(Z_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(z_i-\zeta)^2}{2\sigma^2}} \tag{2}
\]

\[
\ln(L_c(\theta \mid z)) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Z_i - \zeta)^2 \tag{3}
\]

\[
Q(\theta \mid \theta_0, \bar{z}) = \mathbb{E}_{\theta_0}\{\ln L_c(\theta \mid \bar{z}) \mid \theta_0, \bar{z}\} = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \mathbb{E}\{(Z_i - \zeta)^2 \mid \theta_0, \bar{z}\} \tag{4}
\]

(c) Remember that the EM sequence is generated by maximizing the previous expectation. So substitute \( \hat{\zeta}_{(j)} \) and \( \hat{\sigma}^2_{(j)} \) into \( \theta_0 \), and then the MLEs are the conditional expectations given on page 224.