We need to evaluate \( P(x > a) = \int_{a}^{\infty} f(x) \, dx \), \( X \sim f \).

a) We have the following estimators:

\[
\delta_1 = \frac{1}{n} \sum_{i=1}^{n} I(X_i > a), X_i \sim f, \text{iid}
\]

\[
\delta_3 = \frac{1}{n} \sum_{i=1}^{n} I(X_i > \mu),
\]

where \( P(X > \mu) \) is known, and \( a > \mu \).

The control variate estimator:

\[
\delta_2 = \frac{1}{n} \sum_{i=1}^{n} I(X_i > a) + \beta \left[ \frac{1}{n} \sum_{i=1}^{n} I(X_i > \mu) - P(X > \mu) \right].
\]

Since \( \text{var}(\delta_2) = \text{var}(\delta_1) + \beta^2 \text{var}(\delta_3) + 2 \beta \text{cov}(\delta_1, \delta_3) \), and

\[
\text{cov}(\delta_1, \delta_3) = \frac{1}{n} P(X > a)[1 - P(X > \mu)],
\]

we can find \( \text{var}(\delta_3) \):

\[
\text{var}(\delta_3) = \frac{1}{n} \sum_{i=1}^{n} \text{var}(I(X_i > \mu)),
\]

(because \( X_i \) are iid).

\[
\text{var} I(X_i > \mu) = E[I(X_i > \mu)]^2 - [E(I(X_i > \mu))]^2 =
\]

\[
= E[I(X_i > \mu)] - [E(I(X_i > \mu))]^2 = \int_{\mu}^{\infty} f(x) \, dx - \left[ \int_{\mu}^{\infty} f(x) \, dx \right]^2 =
\]

\[
= P(X > \mu) - [P(X > \mu)]^2 = P(X > \mu)[1 - P(X > \mu)]
\]

(We used the fact that \( E[I(X_i > \mu)] = E[I(X_i > \mu)] \).)

So,

\[
\text{var}(\delta_3) = \frac{1}{n^2} n P(x > \mu)[1 - P(x > \mu)] = \frac{1}{n} P(x > \mu)[1 - P(x > \mu)].
\]

b) \( \text{var}(\delta_2) = \text{var}(\delta_1) + \beta^2 \text{var}(\delta_3) + 2 \beta \text{cov}(\delta_1, \delta_3) \).

If we want \( \delta_2 \) to improve \( \delta_1 \) (in the sense of reducing variance), we need the following condition to be satisfied:

\[
\beta^2 \text{var}(\delta_3) + 2 \beta \text{cov}(\delta_1, \delta_3) < 0.
\]
Since $\beta^2 \text{var}(\delta) \geq 0$, and $\text{cov}(\delta_1, \delta_2) \geq 0$ (see part a), we need $\beta < 0$.

Also from the inequality $\beta^2 \text{var}(\delta_3) + 2\beta \text{cov}(\delta_1, \delta_3) < 0$ we get

$$\text{cov}(\delta_1, \delta_3) < \frac{-\beta}{2} \frac{\text{var}(\delta_1)}{\text{var}(\delta_3)}$$

or $|\beta| < 2 \frac{\text{cov}(\delta_1, \delta_3)}{\text{var}(\delta_3)}$, since $\beta < 0$.

So, the conditions on $\beta$ are:

$$\beta < 0 \text{ and } |\beta| < 2 \frac{\text{cov}(\delta_1, \delta_3)}{\text{var}(\delta_3)}.\]

c) $f = N(0,1)$, find $P(X > a)$ for $a = 3, 5, 7$.

After generating 1000000 random variables from $N(0,1)$ and applying the estimator $\delta_1$, we get:

<table>
<thead>
<tr>
<th>a</th>
<th>estimator</th>
</tr>
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<tbody>
<tr>
<td>3</td>
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</tr>
<tr>
<td>5</td>
<td>0.00000122</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
</tr>
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