Monte Carlo Statistical Methods
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Assignment 1 # 1.29

(a) Given that the interval is to be minimized by increasing the average height, the shortest interval will contain the modal value. This area is given by:

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

where \( b \) is a function of \( 'a' \). We need to minimize the range \( R = b(a) - a \).

\[ F(b(a)) - F(a) = 1 - \alpha \]

\[ \Rightarrow F'(b(a))b'(a) - F'(a) = 0 \]

\[ \Rightarrow f(b(a))(1) - f(a) = 0, \text{ since derivative of } R \text{ is 0 at minimum.} \]

So, \( f(b) = f(a) \).

(b) If \( f \) is symmetric, then since \( f(x) = f(-x) \) for all \( x \) and \( f(x) = f(y) \) implies \( x = -y \) or \( y \), then \( f(a) = f(b) \) implies \( a = -b \) or \( b \). But since \( a \) and \( b \) are at opposite ends of the interval then \( a = -b \).

(c) From problem 1.28

(a) \( X/\sigma \sim N(0, \sigma^2) \) and \( 1/\sigma^2 \sim \Gamma(1, 2) \)

Let \( w = 1/\sigma^2 \) so \( f(w) = 1/(\Gamma(1)2)w^{1/2}e^{-w/2} = 1/2e^{-w/2} \)

\( \sigma = 1/w^{1/2} \) so \( dw/\sigma = -2/\sigma^3 \)

By transformation \( f_\sigma(\sigma) = 1/\sigma^3 e^{-1/2\sigma^2} \)

So \( \pi(\sigma|x) = \int_0^\infty 1/\sqrt{2\pi\sigma} e^{-x^2/2\sigma^2} \frac{e^{-1/2\sigma^2}}{\sigma^3} d\sigma = \frac{8(\sigma^2 + 1)^{3/2} e^{-(x^2 + 1)/2\sigma^2}}{\sqrt{\pi} \sigma^3} \)

To find 90% highest posterior credible region we simultaneously solve for \( l \) and \( u \):

\( \pi(\sigma = u/x) = \pi(\sigma = l/x) \) and \( \int \pi(\sigma|x) \, d\sigma = 0.9 \) for limits \( l \) and \( u \).

(c)(b) Similarly, \( \pi(\lambda|x) = \int_0^\infty \frac{x!}{\lambda^{x+1}} e^{-2\lambda} d\lambda \)

\[ e^{-2\lambda} \frac{x!}{\lambda^{x+1}} \frac{e^{-2\lambda}}{x!} = \frac{e^{-2\lambda} \lambda^{x+1} 2^{x+2}}{(x+1)!} \]

To find 90% highest posterior credible region we solve the simultaneous equations for \( u = u(x) \) and \( l = l(x) \).

\( \pi(\lambda = l/x) = \pi(\lambda = u/x) \) and \( \int \pi(\lambda|x) \, d\lambda = 0.9 \) for limits \( l \) and \( u \).

The integral is equivalent to \( e^{-2l} \sum_{n=0}^x \frac{(2l)^n}{n!} - e^{-2u} \sum_{n=0}^x \frac{(2u)^n}{n!} = 0.9 \)
(c) $\pi(p/x) = \frac{(1/\pi) p^{-1/2} (1-p)^{-1/2} \binom{10 + x + 1}{x} p^{10} (1-p)^{x}}{\int_0^1 (1/\pi) \frac{11 + x}{x} p^{9.5} (1-p)^{x-0.5} dp}

\frac{(x+10)!}{\Gamma(10.5)\Gamma(x+0.5)} p^{9.5} (1-p)^{(x-0.5)}$

The 90% highest posterior credible region is given by solving the simultaneous equations $\pi(p=1/x) = \pi(p=u/x)$ and the integral of $\pi(p/x)$ within the interval 1 to u.