Consider $n$ observations $x_1, ..., x_n$ from $B$ where both $k$ and $p$ are unknown.

(a) Show that the maximum likelihood estimator of $k$, $\hat{k}$, satisfies

$$\left( \hat{k}(1 - \hat{p}) \right)^n \geq \prod_{i=1}^{n} (\hat{k} - x_i)$$

and

$$\left( (\hat{k} + 1)(1 - \hat{p}) \right)^n < \prod_{i=1}^{n} (\hat{k} + 1 - x_i),$$

where $\hat{p}$ is the maximum likelihood estimator of $p$.

Density function of Binomial Distribution $B(n, p)(0 \geq p \geq 1)$ is

$$f(x|p) = \binom{k}{x} p^x (1 - p)^{k-x}.$$

The likelihood function for the observations $x_1, ..., x_n$ is

$$L(k, p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \binom{k}{x_i} p^{x_i} (1 - p)^{(k-x_i)}$$

For maximum $(\hat{k}, \hat{p})$ of $L(k, p)$ the following should be valid:

$$L(\hat{k} - 1, \hat{p}) \leq L(\hat{k}, \hat{p}) \geq L(\hat{k} + 1, \hat{p})$$

Taking into account that

$$L(\hat{k} - 1, \hat{p}) = \prod_{i=1}^{n} \frac{\hat{k}!}{x_i! (\hat{k} - x_i)!} \hat{p}^{x_i} (1 - \hat{p})^{(\hat{k} - x_i)},$$

$$L(\hat{k}, \hat{p}) = \prod_{i=1}^{n} \frac{k!}{x_i! (k - x_i)!} \hat{p}^{x_i} (1 - \hat{p})^{(k - x_i)}$$

$$= \prod_{i=1}^{n} \frac{k}{(k-x_i)} (1 - \hat{p}) \prod_{i=1}^{n} \frac{\hat{k}!}{x_i! (\hat{k}-x_i)!} \hat{p}^{x_i} (1 - \hat{p})^{(\hat{k}-x_i)},$$

$$L(\hat{k}, \hat{p}) = \prod_{i=1}^{n} \frac{(\hat{k}+1)!}{x_i! (\hat{k}+1 - x_i)!} \hat{p}^{x_i} (1 - \hat{p})^{(\hat{k}+1 - x_i)}$$

$$= \prod_{i=1}^{n} \frac{(\hat{k}+1)}{(\hat{k}+1-x_i)} (1 - \hat{p}) \prod_{i=1}^{n} \frac{\hat{k}!}{x_i! (\hat{k}-x_i)!} \hat{p}^{x_i} (1 - \hat{p})^{(\hat{k}-x_i)},$$

we come to:

$$1 \leq \prod_{i=1}^{n} \frac{\hat{k}}{(k-x_i)} (1 - \hat{p}),$$

and

$$1 \geq \prod_{i=1}^{n} \frac{\hat{k}+1}{(k+1-x_i)} (1 - \hat{p}).$$
After transformation we get:

\[(\hat{k}(1 - \hat{p}))^n \geq \prod_{i=1}^{n}(\hat{k} - x_i),\]

and

\[((\hat{k} + 1)(1 - \hat{p}))^n \leq \prod_{i=1}^{n}(\hat{k} + 1 - x_i).\]

Without losses of generality we can assume that the last inequality validates as strict. The equalities (1) and (2) are proved.

We will also need an expression connecting \(\hat{p}\) and \(\hat{k}\). To find it we solve the following equation:

\[
\frac{\partial}{\partial p} \ln(L(k, p)) = 0;
\]

\[
\frac{\partial}{\partial \hat{p}} \left( \ln \left( \prod_{i=1}^{n} \frac{\hat{k}^x}{x!(\hat{k} - x)!} \right) + \ln \left( \prod_{i=1}^{n} \hat{p}^{x_i} (1 - \hat{p})^{(\hat{k} - x_i)} \right) \right) = 0;
\]

\[
\sum_{i=1}^{n} \left( \frac{x_i}{\hat{p}} - (\hat{k} - x_i) \frac{1}{1 - \hat{p}} \right) = 0;
\]

\[
\frac{\sum_{i=1}^{n} x_i}{\hat{p}(1 - \hat{p})} - \frac{n\hat{k}}{1 - \hat{p}} = 0;
\]

\[
\hat{p} = \frac{\sum_{i=1}^{n} x_i}{kn};
\]

(b) If the sample is 16, 18, 22, 25, 27, show that \(\hat{k} = 99\).

Compute logarithms for both sides of equalities (1) and (2):

<table>
<thead>
<tr>
<th>(k)</th>
<th>97</th>
<th>98</th>
<th>99</th>
<th>100</th>
<th>101</th>
<th>190</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ln(\prod_{i=1}^{n}(k - x_i)))</td>
<td>21.607</td>
<td>21.673</td>
<td>21.738</td>
<td>21.802</td>
<td>21.866</td>
<td>25.630</td>
</tr>
<tr>
<td>(\ln(\prod_{i=1}^{n}(k + 1 - x_i)))</td>
<td>21.673</td>
<td>21.738</td>
<td>21.802</td>
<td>21.866</td>
<td>21.928</td>
<td>25.660</td>
</tr>
</tbody>
</table>

Assuming that the likelihood function is unimodal we find \(\hat{k} = 99\).

(c) If the sample is 16, 18, 22, 25, 28, show that \(\hat{k} = 190\).

Compute logarithms for both sides of equalities (1) and (2):

<table>
<thead>
<tr>
<th>(k)</th>
<th>99</th>
<th>188</th>
<th>189</th>
<th>190</th>
<th>191</th>
<th>192</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ln((k(1 - \hat{p}))^n))</td>
<td>22.366</td>
<td>25.573</td>
<td>25.599</td>
<td>25.626</td>
<td>25.652</td>
<td>25.678</td>
</tr>
<tr>
<td>(\ln(\prod_{i=1}^{n}(k - x_i)))</td>
<td>21.724</td>
<td>25.564</td>
<td>25.594</td>
<td>25.624</td>
<td>25.654</td>
<td>25.683</td>
</tr>
<tr>
<td>(\ln(((k + 1)(1 - \hat{p}))^n))</td>
<td>22.416</td>
<td>25.599</td>
<td>25.626</td>
<td>25.652</td>
<td>25.678</td>
<td>25.704</td>
</tr>
<tr>
<td>(\ln(\prod_{i=1}^{n}(k + 1 - x_i)))</td>
<td>21.788</td>
<td>25.594</td>
<td>25.624</td>
<td>25.654</td>
<td>25.683</td>
<td>25.713</td>
</tr>
</tbody>
</table>

Assuming that the likelihood function is unimodal we find \(\hat{k} = 190\).

Conclusion: Maximum Likelihood Estimator is not robust in the presence of errors. The example demonstrates that a small deviation in data can increase an estimation more than two times. For more details the reader can be addressed to the following article: