Correlated Random Walks

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In a random walk, a subject takes steps at regular intervals. The direction is chosen at random, but some directions may be more likely than others.

Here is a random walk in which all steps are 1 unit in length, and all directions are equally likely:

![Random Walk Diagram](image)

Figure 1: Example of a random walk. Start at origin.
Here is a random walk in which all directions are equally likely, and all step lengths between 0 and 1 are equally likely:

Figure 2: Example of a random walk, variable step length. Start at origin.

Directions may not all be equally likely. Here is a probability distribution on the circle (that is, on $[-\pi, \pi]$ that is not uniform:

Figure 3: Nonuniform distribution on $[-\pi, \pi]$. 
Here is a random walk in which step lengths may be anything from 0 to 1, but tend to be near 0.5. Directions are chosen from the distribution on $[-\pi, \pi]$.

Figure 4: Random walk, directions chosen at random, from nonuniform distribution. Start at origin.

Notice a tendency to drift right, since circular distribution puts most weight around angle $\theta = 0$.

In all these random walks, choice of direction at step $i$ is unrelated to direction of previous step.

In much animal motion, however, direction is persistent. The animal changes its direction gradually from step to step. This is commonly modeled with correlated random walk.

In correlated random walk, direction chosen at step $i+1$ is related to direction of step $i$. What is randomly chosen is the change in direction from step $i$ to step $i+1$. This change in direction is the turning angle for step $i+1$. See Kareiva & Shigesada (1983) or McCulloch & Cain (1989).
Suppose step \( i \) has direction \( \theta_i \). Choose an angle \( \delta \theta \) randomly. Then let step \( i + 1 \) have direction \( \theta_{i+1} = \theta_i + \delta \theta \). If we choose the turning angles from a distribution that is concentrated about 0, there will usually be little change in direction from one step to the next. Example:

![Figure 5: Correlated random walk. Start at origin.](image-url)

Notice that this path for the most part is almost straight or gently curving. It is not nearly as jagged as the previous paths, and does not cross itself often.
We can model other sorts of paths by changing the parameters of the random walk. Here is a distribution on the circle which is concentrated about 0.2:

![Distribution on circle, nonzero mean.](image)

Figure 6: Distribution on circle, nonzero mean.

If we draw turning angles from this distribution, we can get a path such as this:

![Random walk, turning angles from distribution above. Start at origin.](image)

Figure 7: Random walk, turning angles from distribution above. Start at origin.

We have modeled a subject that tends to turn slightly to the left (think of a three-legged turtle).
Lévy Flights

Lévy flights are mentioned in the paper by Mårell, Ball, & Hofgaard as an alternative to correlated random walks.

A Lévy flight is a random walk in which the step lengths are drawn from a heavy-tailed distribution. A heavy-tailed distribution goes to zero, but so slowly that it does not have finite variance. An example is the Cauchy distribution:

\[
f(x) = \frac{1}{\pi(1+x^2)}.
\]

The probability that a step length of less than \( k \) is drawn is \( 2\arctan k \). Most steps will be short, but once in a while a very long step will occur.

Step directions are chosen as in the ordinary random walk. All directions are equally likely.
Example:

This path seems to have an overall direction, simply because one step is so much longer than all others. Also, a Lévy flight will tend to form clusters – sequences of short steps, separated by long steps. Many animals tend to forage in a small area, then move quickly over some distance to another patch and start foraging again. The reindeer in the paper by Märell et. al. did this. Lévy flights may be more suited to modeling such irregular foraging than correlated random walks.