Chapter 1: Probability in the World Around Us

1.1 Why Study Probability?

We live in an information society. We are confronted—in fact, inundated—with quantitative information at all levels of endeavor. Charts, graphs, rates, percentages, averages, forecasts, and trend lines are an inescapable part of our everyday lives. They affect small decisions we make every day, such as whether to wear a coat or to take an umbrella when we leave the house in the morning. They affect larger decisions on health, citizenship, parenthood, jobs, financial concerns, and many other important matters. Today, an informed person must have some facility for dealing with data and making intelligent decisions based on quantitative arguments that involve uncertainty or chance.

We live in a scientific age. We are confronted with arguments that demand logical, scientific reasoning, even if we are not trained scientists. We must be able to make our way successfully through a maze of reported “facts,” in order to separate credible conclusions from specious ones. We must be able to weigh intelligently such issues as the evidence on the causes of cancer, the effects of pollutants on the environment, and the likely results of eating genetically modified plants and animals.

We live amidst burgeoning technology. We are confronted with a job market that demands scientific and technological skills; and students must be trained to deal with the tools of this technology productively, efficiently, and correctly. Much of this new technology is concerned with information processing and dissemination, and proper use of this technology requires probabilistic skills. These skills are in demand in engineering, business, and computer science for jobs involving market research, product development and testing, economic forecasting, credit research, quality control, reliability, business management, and data management, to name just a few.

Few results in the natural or social sciences are known absolutely. Most are reported in terms of chances or probabilities: the chance of rain tomorrow, the chance of your getting home from school or work safely, the chance of your living past sixty years of age, the chance of contracting (or recovering from) a certain disease, the chance of inheriting a certain trait, the chance of your annual income exceeding $40,000 in two years, the chance of winning an election. Today’s adults must obtain some knowledge of probability and must be able to tie probabilistic concepts to real scientific investigations if they are to understand science and the world around them.

This book provides an introduction to probability that is both mathematical, in the sense that the underlying theory is developed from axioms, and practical, in the sense that applications to real-world problems are discussed. The material is designed to provide a strong basis in probability for students who may go on to deeper studies of statistics, mathematics, engineering, business, or the physical and biological sciences; at the same time, it should provide a basis for practical decision making in the face of uncertainty.
1.2 Deterministic and Probabilistic Models

1.2.1 Modeling Reality

It is essential that we grasp the difference between theory and reality. Theories are ideas proposed to explain phenomena in the real world. As such, they are approximations or models of reality. Sometimes, theories are wrong. For example, prior to Copernicus, scientists believed the theory that the Sun and other planets revolved around the Earth. Copernicus was the first to recognize that the Earth and other planets revolved around the Sun, but he believed that the orbits of the planets were circular. Thus, Copernicus’s theory (or model) of the solar system was closer to, but not the same as, reality. Through the scientific process, theories are constantly refined so that they become closer to reality.

Theories are presented in verbal form in some (less quantitative) fields and as mathematical relationships in others. Thus, a theory of social change might be expressed verbally in sociology, whereas the theory of heat transfer is presented in a precise and deterministic mathematical manner in physics. Neither gives an accurate and unerring explanation for real life, however. Slight variations from the mathematically expected can be observed in heat-transfer phenomena and in other areas of physics. The deviations cannot be blamed solely on the measuring instruments (the explanation that one often hears); they are due in part to a lack of agreement between theory and reality. These differences between theory and reality led George Box (1979, p. 202), a famous statistician, to note: “All models are wrong, but some are useful.”

In this text, we shall develop certain theoretical models of reality. We shall attempt to explain the motivation behind such a development and the uses of the resulting models. At the outset, we shall discuss the nature and importance of model building in the real world, to convey a clear idea of the meaning of the term model and of the types of models generally encountered.

1.2.2 Deterministic Models

Suppose that we wish to measure the area covered by a lake that, for all practical purposes, has a circular shoreline. Since we know that the area $A$ is given by $A = \pi r^2$, where $r$ is the radius, we attempt to measure the radius (perhaps by averaging a number of measurements taken at various points), and then we substitute the value obtained into the formula. The formula $A = \pi r^2$, as used here, constitutes a deterministic model. It is deterministic because, once the radius is known, the area is assumed to be known. It is a model of reality because the true border of the lake has some irregularities and therefore does not form a true circle. Even though the planar object in question is not exactly a circle, the model identifies a useful relationship between the area and the radius, which makes approximate measurements of area easy to calculate. Of course, the model becomes poorer and poorer as the shape of the figure deviates more and more from that of a circle until, eventually, it ceases to be of value and a new model must take over.
Another deterministic model is Ohm’s Law, $I = E/R$, which states that electric current is directly proportional to the voltage and inversely proportional to the resistance in a circuit. Once the voltage and the resistance are known, the current can be determined. If we investigated many circuits with identical voltages and resistances, we might find that the current measurements differed by small amounts from circuit to circuit, owing to inaccuracies in the measuring equipment or other uncontrolled influences. Nevertheless, any such discrepancies are negligible, and Ohm’s Law thus provides a useful deterministic model of reality.

1.2.3 Probabilistic Models

Contrast the two preceding situations with the problem of tossing a balanced coin and observing the upper face. No matter how many measurements we may make on the coin before it is tossed, we cannot predict with absolute accuracy whether the coin will come up heads or tails. However, it is reasonable to assume that, if many identical tosses are made, approximately one-half will result in outcomes of heads; that is, we cannot predict the outcomes of the next toss, but we can predict what will happen in the long run. We sometimes convey this long-run information by saying that the “chance” or “probability” of heads on a single toss is $\frac{1}{2}$. This probability statement is actually a formulation of a probabilistic model of reality. Probabilistic models are useful in describing experiments that give rise to random, or chance, outcomes. In some situations, such as the tossing of an unbalanced coin, preliminary experimentation must be conducted before realistic probabilities can be assigned to the outcomes; but it is possible to construct fairly accurate probabilistic models for many real-world phenomena. Such models are useful in varied applications, such as in describing the movement of particles in physics (Brownian motion), in explaining the changes in the deer population within a region, and in predicting the profits for a corporation during some future quarter.

1.3 Applications in Probability

We shall now consider two uses of probability theory. Both involve an underlying probabilistic model, but the first hypothesizes a model and then uses this model for practical purposes, whereas the second deals with the more basic question of whether the hypothesized model is in fact a correct one.

Suppose that we attempt to model the random behavior of the arrival times and lengths of service for patients at a medical clinic. Such a mathematical function would be useful in describing the physical layout of the building and in helping us determine how many physicians are needed to service the facility. Thus, this use of probability assumes that the probabilistic model is known and offers a good characterization of the real system. The model is then employed to enable us to infer the behavior of one or more variables. The inferences will be correct—or nearly correct—if the assumptions that governed construction of the model were correct.

The problem of choosing the correct model introduces the second use of probability theory, and this use reverses the reasoning procedure just described. Assume
that we do not know the probabilistic mechanism governing the behavior of arrival and service times at the clinic. We might then observe an operating clinic and acquire a sample of arrival and service times. Based on the sample data, inferences can be drawn about the nature of the underlying probabilistic mechanism—a type of application known as *statistical inference*. This book deals mostly with problems of the first type but, on occasion, it makes use of data as a basis for model formulation. Ideally, readers will go on to take a formal course in statistical inference later in their academic studies.

Consider the problem of replacing the light bulbs in a particular socket in a factory. A bulb is to be replaced either at failure or at a specific age $T$, whichever comes first. Suppose that the cost $c_1$ of replacing a failed bulb is greater than the cost $c_2$ of replacing a bulb at age $T$. This may be true because in-service failures disrupt the factory, whereas scheduled replacements do not. A simple probabilistic model enables us to conclude that the average replacement cost $C_a$ per unit time, in the long run, is approximately

$$C_a = \frac{1}{\mu} \left[ c_1 \left( \text{Probability of an in-service failure} \right) + c_2 \left( \text{Probability of a planned replacement} \right) \right]$$

where $\mu$ denotes the average service time per bulb. The average cost is a function of $T$; and if $\mu$ and the indicated probabilities can be obtained from the model, a value of $T$ can be chosen to minimize this function. Problems of this type are discussed more fully in Chapter 9.

Biological populations are often characterized by birth rates, death rates, and a probabilistic model that relates the size of the population at a given time to these rates. One simple model allows us to show that a population has a high probability of becoming extinct even if the birth and death rates are equal. Only if the birth rate exceeds the death rate might the population exist indefinitely.

Again referring to biological populations, models have been developed to explain the diffusion of a population across a geographic area. One such model concludes that the square root of the area covered by a population is linearly related to the length of time the population has been in existence. This relationship has been shown to hold reasonably well for many varieties of plants and animals.

Probabilistic models like those mentioned give scientists a wealth of information for explaining and controlling natural phenomena. Much of this information is intuitively clear, such as the fact that connecting identical components in series reduces the system’s expected life length compared to that of a single component, whereas parallel connections increase the system’s expected life length. But many results of parallel connections increase the system’s expected life length. But many results of probabilistic models offer new insights into natural phenomena—such as the fact that, if a person has a 50:50 chance of winning on any one trial of a gambling game, the excess of wins over losses will tend to stay either positive or negative for long periods of time, given that the
trials are independent. (That is, the difference between number of wins and number of losses does not fluctuate rapidly from positive to negative.)

1.4 A Brief Historical Note

The study of probability has its origins in games of chance, which have been played throughout recorded history and, no doubt, during prehistoric times as well. The astragalus, a bone that lies above the talus (heel bone), was used in various board games in Egypt (c. 3500 B.C.). A game called “hounds and jackals” by the excavators of Egyptian tombs apparently used astragali in the same manner as dice. The hounds and jackals were moved on the board according to the results of throwing the astragali. Homer (c. 900) reported that Patroclus became so angry with his opponent while playing a game based on astragali that he nearly killed him (David, 1955; Folk, 1981).

The Ancient Greeks used the knucklebones of sheep or goats, as illustrated in Figure 1.1, to make astragali for games of chance. The Romans used the term “tali”, which is the Latin name for knucklebones. They made tali from brass, silver, gold, ivory, marble, wood, bone, bronze, glass, terracotta, and precious gems. When tossed, the astragali, or tali, would land on one of four sides. The most popular game resembled modern dice.

Figure 1.1: Astragali as used in Ancient Greece

![Astragali as used in Ancient Greece](http://www.personal.psu.edu/wxk116/roma/tali.html)

Gaming was so popular in Roman times that laws had to be passed to regulate it. The Church’s stern opposition followed and continues today. Yet, gaming and gambling have flourished among all classes of people. Most early games involved the astragalus. Dice and cards were used later. Dice dating from the beginning of the third millennium are the earliest found so far. A die made from well-fired buff potter and found in Iraq has the opposite points in consecutive order: 2 opposite 3, 4 opposite 5, and 6 opposite 1 (Figure 1.2). Die with opposite faces totaling 7 must have evolved about 1400 B.C. (David, 1955; Folks, 1981).
In addition to the use of games, chance mechanisms have been used to divine the will of the gods. In 1737, John Wesley sought guidance by drawing lots to decide whether or not to marry. Even today, a group will sometimes draw straws with the agreement that the person drawing the short straw will do the unpleasant task at hand.

In 1494, Fra Luca Pacciolo published a mathematical discussion of the problem of points, that is, the problem of how to divide equitably the stakes between two players when a game is interrupted before its conclusion. However, it is generally agreed that a major impetus to the formal study of probability was provided by the Chevalier de Méré when he posed a problem of points to the famous mathematician Blaise Pascal (1623-1662). The question was along the following lines. To win a particular game of chance, a gambler must throw a 6 with a die; he has eight throws in which to do it. If he has no success on the first three throws, and the game is thereupon ended prematurely, how much of the stake is rightfully his? Pascal cast this problem in probabilistic terms and engaged in extensive correspondence with another French mathematician, Pierre de Fermat (1608-1665), about its solution. This correspondence began the formal mathematical development of probability theory.

Scientists in the eighteenth century (such as James Bernoulli and Abraham de Moivre) continued to develop probability theory and recognized its usefulness in solving important problems in science. The normal curve was introduced during this period. Their work was carried forward by Carl Friedrich Gauss and Pierre de Laplace in the nineteenth century, when the use of probability in data analysis emerged as a forerunner of modern statistics. In the twentieth century, with the work of Kolmogorov, probability theory became a major branch of mathematical research; and applications of the theory have spread to virtually every corner of scientific research.
1.5 A Look Ahead

This text is concerned with the theory and applications of probability as a model of reality. We shall postulate theoretical frequency distributions for populations and develop a theory of probability in a precise mathematical manner. The net result will be a theoretical or mathematical model for acquiring and utilizing information in real life. It will not be an exact representation of nature, but this should not disturb us. Like other theories, its utility should be gauged by its ability to assist us in understanding nature and in solving problems in the real world. Such is the role of the theory of heat transfer, the theory of strengths of materials, and other models of nature.

References

