STA 6329 – EXAM 2 Practice Problems – Part 2

17 $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 6 & -2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $\underset{\sim}{X} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 58 \\ -27 \\ 70 \\ 63 \end{bmatrix}$

Want rows 1, 2, 4, 3 Columns 1, 2, 3, 4

$\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $Q = I$

$PA\underset{\sim}{X} = P\underset{\sim}{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 58 \\ -27 \\ 63 \\ 70 \end{bmatrix}$

$A_{11}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A_{12}^* = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ $A_{21}^* = [3 \ \ 6 \ \ -2]$ $A_{22}^* = [4]$

$X_1^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $X_2^* = [x_4]$ $B_1^* = \begin{bmatrix} 58 \\ -27 \\ 63 \end{bmatrix}$ $B_2^* = [70]$

$A_{11}^* C = B_1^* = C = B_1^* = \begin{bmatrix} 58 \\ -27 \\ 63 \end{bmatrix}$ $A_{11}^* K = A_{12}^* \Rightarrow K = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$A_{22}^* - A_{21}^* K = 4 - [3(2) + 6(-1) + (-2)(2)] = 8$

$B_2^* - A_{21}^* C = 70 - [3(58) + 6(-27) + (-2)(63)] = 70 - (-114) = 184$

$X_2^* = (8)^{-1}(184) = 23$ $X_1^* = C - KX_2^* = \begin{bmatrix} 58 \\ -27 \\ 63 \end{bmatrix} - 23 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \\ 17 \end{bmatrix}$

Check: $X = \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \\ 17 \\ 23 \end{bmatrix} \Rightarrow AX = \begin{bmatrix} 12 + 46 \\ -4 - 23 \\ 36 - 24 - 34 + 92 \\ 17 + 46 \end{bmatrix} = \begin{bmatrix} 58 \\ -27 \\ 70 \\ 63 \end{bmatrix} = \underset{\sim}{b}$ ✓

18. $X_0 \equiv$ a particular solution to $AXC = B$ $X_0 = A^- BC^-$

Let: $X^* = X_0 + A^-AR(I - CC^-) + (I - A^-A)SCC^- + (I - A^-A)T(I - CC^-)$

for arbitrary R, S, T of appropriate dimensions.

$$AX^*C = AX_0C + AA^-AR(I - CC^-)C + A(I - A^-A)SCC^-C$$
$$+ A(I - A^-A)T(I - CC^-)C$$

$$= AA^-BC^-C + AR(C - C) + (A - A)SC + (A - A)T(C - C)$$

$= AA^-BCC^- \equiv$ Solution to $AXC = B$ since $X_0 \equiv$ solution

19. $Y_1, \ldots, Y_p \equiv$ matrices in linear space $\mathcal{V}$, $\mathcal{U} \subset \mathcal{V}$

Let $Z_1, \ldots, Z_p \equiv$ projections of $Y_1, \ldots, Y_p$ on $\mathcal{U}$ (respectively)

Then for scalars $k_1, \ldots, k_p$, the projection of $\sum_{i=1}^{p} k_i Y_i$ on $\mathcal{U}$ is $\sum_{i=1}^{p} k_i Z_i$

Proof: Since $Z_1, \ldots, Z_p \equiv$ Projections of $Y_1, \ldots, Y_p$ on $\mathcal{U}$ $\sum_i k_i Y_i \in \mathcal{V}$ $\sum_i k_i Z_i \in \mathcal{U}$

$$(Y_i - Z_i) \cdot Z_i = 0 \qquad i = 1, \ldots, p \implies \sum_{i=1}^{p} (Y_i - Z_i) \cdot Z_i = 0$$

Now: $$\sum_{i=1}^{p} k_i (Y_i - Z_i) \cdot Z_i = \sum_{i=1}^{p} k_i \, 0 = 0$$

$$\implies \sum_i k_i (Y_i - Z_i) \perp \mathcal{U} \implies \sum_i k_i Z_i \equiv \text{projection of } \sum_i k_i Y_i \text{ on } \mathcal{U}$$

Better way: Let $X \in \mathcal{U} \implies (Y_i - Z_i) \cdot X = 0$ $(Y_i - Z_i) \perp \mathcal{U}$

$$[(k_1 Y_1 + \cdots + k_p Y_p) - (k_1 Z_1 + \cdots + k_p Z_p)] \cdot X = k_1[(Y_1 - Z_1) \cdot X] + \cdots + k_p[(Y_p - Z_p) \cdot X]$$
$$= k_1(0) + \cdots + k_p(0) = 0 \implies \text{Result}$$

2.3

20 $\underset{\sim}{y} \in \mathbb{R}^3 \equiv V \quad X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathcal{U} \equiv \mathcal{C}(X)$

a) $XC = X[\underset{\sim}{c}_1 \ \underset{\sim}{c}_2]$ Let $\underset{\sim}{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underset{\sim}{c}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$XC = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 7 \\ 3 & 13 \end{bmatrix}$ 2 Points $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 13 \end{bmatrix}$

b) $X(X'X)^{-1}X' \equiv$ projection matrix $\quad X'X = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \quad (X'X)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$

$$X(X'X)^{-1}X' = \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix} = P$$

c) $\underset{\sim}{z}_1 = P\underset{\sim}{y}_1 = \frac{1}{6}\begin{bmatrix} -3 \\ 12 \\ 27 \end{bmatrix} \quad \underset{\sim}{z}_2 = \frac{1}{6}\begin{bmatrix} -11 \\ 4 \\ 19 \end{bmatrix} \quad \underset{\sim}{z}_3 = \frac{1}{6}\begin{bmatrix} 10 \\ 28 \\ 46 \end{bmatrix}$

d) $3\underset{\sim}{y}_1 - \underset{\sim}{y}_2 + 2\underset{\sim}{y}_3 = \begin{bmatrix} 6 \\ 10 \\ 28 \end{bmatrix} \quad P(3\underset{\sim}{y}_1 - \underset{\sim}{y}_2 + 2\underset{\sim}{y}_3) = \frac{1}{6}\begin{bmatrix} 22 \\ 88 \\ 154 \end{bmatrix}$

$$3\underset{\sim}{z}_1 - \underset{\sim}{z}_2 + 2\underset{\sim}{z}_3 = \frac{1}{6}\begin{bmatrix} -9 + 11 + 20 \\ 36 - 4 + 56 \\ 81 - 19 + 92 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 22 \\ 88 \\ 154 \end{bmatrix} \checkmark$$

e) $(y_1 - z_1)'X \quad \begin{bmatrix} 0 - (-1/2) \\ 1 - 2 \\ 5 - 4.5 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$

$(\underset{\sim}{y}_1 - \underset{\sim}{z}_1)'X = \left[\frac{1}{2} - 1 + \frac{1}{2} \ \vdots \ 0 - 1 + 1\right] = [0 \ 0] \checkmark$

2.4

22 $\quad X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\sim}{y} \in \mathbb{R}^6 \qquad \underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 26 \\ 28 \\ 9 \\ 5 \end{bmatrix}$

a) $X'X = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \qquad (X'X)^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

$X'\underset{\sim}{y} = \begin{bmatrix} 102 \\ 34 \\ 54 \\ 14 \end{bmatrix} \qquad \underset{\sim}{b_0} = (X'X)^- X'\underset{\sim}{y} = \begin{bmatrix} 0 \\ 17 \\ 27 \\ 7 \end{bmatrix}$

b) $P_X = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

c) $z = P_X Y = \begin{bmatrix} 17 \\ 17 \\ 27 \\ 27 \\ 7 \\ 7 \end{bmatrix} \qquad \underset{\sim}{y} - \underset{\sim}{z} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \qquad (y-z)'X = [0 \; 0 \; 0 \; 0] = 0$

d) Let $A = I_6 - P_X = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{bmatrix}$

$AX = 0 \Rightarrow A \in \mathcal{C}^{\perp}(X)$

2.5

23. a)

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = X \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow W \in \mathcal{C}(X)$$

b) $$W'W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (W'W)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$W'\underset{\sim}{y} = \begin{bmatrix} 34 \\ 54 \\ 14 \end{bmatrix} \qquad \underset{\sim}{b} = (W'W)^{-1}W'\underset{\sim}{y} = \begin{bmatrix} 17 \\ 27 \\ 7 \end{bmatrix}$$

c)

Numerically $$P_X W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof

$P_X W = X(X'X)^- X'W$

$= X(X'X)^- X'XC$ for C ↑

$= XC = W$ since $(X'X)^- X' \equiv$ G-inverse of X

(Theorem 12.3.4. (1)

d) See 22 b)

24 a) $4! = 24$ 4 ways on row 1 → 3 on row 2 → 2 on row 3 → 1 on row 4

$a_1(1,2,3,4)$

b) $|A| = (-1)6(-1)(-2)(1) = 1(12) = 12$

c) yes $|A| \neq 0$

d)

(2.6)

25 Permutation Matrix is orthogonal $\Rightarrow |P| = \pm 1$ depending on row/col Struktur

$$\phi_n(5,3,2,6,1,4) = 4+2+1+2+0=9 \quad (-1)^9 = -1 \Rightarrow |P| = -1$$

26. a) $A = \begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 4 \\ 1 & -6 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} -1 & -26 \\ 9 & -38 \end{bmatrix}$

b) $|A| = 20-(-14) = 34 \quad |B| = 12-4 = 8 \quad |AB| = 38-(-234)$
$= 272$

$$|A||B| = 34(8) = 272 = |AB|$$

27. $A = \begin{vmatrix} T & O \\ V & W \end{vmatrix} \quad T = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} \quad W = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$$|T| = 8-(-3) = 11 \qquad |W| = 0-(-2) = 2$$

$$\Rightarrow |A| = |T||W| = 11(2) = 22$$

28. $A^2 = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} -64 & -57 \\ 57 & -26 \end{bmatrix} \Rightarrow |A^3| = (-64)(-26) - (-57)(57)$
$= 4913$

$$|A| = 8-(-9) = 17 \Rightarrow 17^3 = 4913 \checkmark$$

29. $A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 1 & 2 \\ 0 & -2 & 3 \end{bmatrix}$

a) Minor of a_{11}: $\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 7$ $\quad a_{12}: \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -3$ $\quad a_{13}: \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} = 2$

Cofactors: $\alpha_{11} = (-1)^{1+1}(7) = 7$ $\quad \alpha_{12} = (-1)^{1+2}(-3) = 3$ $\quad \alpha_{13} = (-1)^{1+3}(2) = 2$

b) $|A| = 2(7) + 3(3) + (-1)(2) = 18$

c) $\sum_{j=1}^{3} a_{2j}\alpha_{1j} = (-1)(7) + 1(3) + 2(2) = -7 + 3 + 4 = 0$

30. $A_1 = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$ $\quad B_1 = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$ $\quad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^2$

$B_2 = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$

a) $\underset{\sim}{x}' A_1 \underset{\sim}{x} = 4x_1^2 - x_1x_2 + 2x_2^2$ $\qquad \underset{\sim}{x}' B_1 \underset{\sim}{x} = 4x_1^2 - x_1x_2 + 2x_2^2$

$\underset{\sim}{x}' B_2 \underset{\sim}{x} = 4x_1^2 - x_1x_2 + 2x_2^2$

b) yes (Lemma 14.1.2.)

c) i) QR $\quad \underset{\sim}{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\quad x_{12} = \frac{\underset{\sim}{a}_2'\underset{\sim}{b}_1}{\underset{\sim}{b}_1'\underset{\sim}{b}_1} = \frac{-2(4)+2(1)}{4^2+1^2} = \frac{-6}{17}$

$\Rightarrow \underset{\sim}{b}_2 = \underset{\sim}{a}_2 - x_{12}\underset{\sim}{b}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} - \left(-\frac{6}{17}\right)\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -10/17 \\ 40/17 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 4 & -10/17 \\ 1 & 40/17 \end{bmatrix}$

$\|\underset{\sim}{b}_1\| = \sqrt{17}$ $\quad \|\underset{\sim}{b}_2\| = \sqrt{\frac{1700}{289}} = \frac{10}{\sqrt{17}}$ $\quad X = \begin{bmatrix} 1 & -6/17 \\ 0 & 1 \end{bmatrix} \rightarrow$

30. c) i) continued $\quad E = \begin{bmatrix} \sqrt{17} & 0 \\ 0 & 10/\sqrt{17} \end{bmatrix} \quad D = \begin{bmatrix} 1/\sqrt{17} & 0 \\ 0 & \sqrt{17}/10 \end{bmatrix}$

$$Q = BD = \begin{bmatrix} 4 & -10/17 \\ 1 & 40/17 \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} & 0 \\ 0 & \sqrt{17}/10 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{17} & -1/\sqrt{17} \\ \frac{1}{\sqrt{17}} & 4/\sqrt{17} \end{bmatrix}$$

Check: $QQ' = \begin{bmatrix} 4/\sqrt{17} & -1/\sqrt{17} \\ \frac{1}{\sqrt{17}} & 4/\sqrt{17} \end{bmatrix} \begin{bmatrix} 4/\sqrt{17} & 1/\sqrt{17} \\ -1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$$R = EX = \begin{bmatrix} \sqrt{17} & 0 \\ 0 & 10/\sqrt{17} \end{bmatrix} \begin{bmatrix} 1 & -6/17 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{17} & -6/\sqrt{17} \\ 0 & 10/\sqrt{17} \end{bmatrix}$$

Check: $QR = \begin{bmatrix} 4/\sqrt{17} & -1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix} \begin{bmatrix} \sqrt{17} & -6/\sqrt{17} \\ 0 & 10/\sqrt{17} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$ ✓

ii) LU Decomposition $\quad L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

$$A = LU = \begin{bmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{bmatrix} \Rightarrow u_{11} = 4 \quad u_{12} = -2$$

$$l_{21}u_{11} = 1 = l_{21}(4) \Rightarrow l_{21} = \frac{1}{4}$$

$$l_{21}u_{12} + u_{22} = 2 = \frac{1}{4}(-2) + u_{22} \Rightarrow u_{22} = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 \\ 1/4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & -2 \\ 0 & 5/2 \end{bmatrix}$$

Check $LU = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$ ✓

iii), iv) Cholesky – A not symmetric ⇒ not positive definite (according to some authors)
Eigenvalues are complex