(1) Consider the full rank matrix $A = \begin{bmatrix} \underset{m\times m}{T} & \underset{m\times n}{U} \\ \underset{n\times m}{V} & \underset{m\times m}{W} \end{bmatrix}$, $B = \begin{bmatrix} T & U \\ V & W \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

where the dimensions of $B_{11}, B_{12}, B_{21}, B_{22}$ are same as T, U, V, W.

Assuming $T, W \equiv$ nonsingular $\Rightarrow$ $B_{11}, B_{22} \equiv$ nonsingular. Let $Q = W - VT^{-1}U$

Show that $B_{11} = T^{-1} + T^{-1}UQ^{-1}VT^{-1}$

$B_{12} = -T^{-1}UQ^{-1}$ $\quad B_{21} = -Q^{-1}VT^{-1}$ $\quad B_{22} = Q^{-1}$

why showing that $AB = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

(2) Consider the unit upper Triangular matrix $A = \begin{bmatrix} 1 & 2 & 7 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

obtain A^{-1}

(3) Show that the matrix $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

(4) Let $\underset{2\times 3}{A} = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 3 \end{bmatrix}$

a) what is the rank (A) ?

b) Give the right matrix $\underset{3\times 2}{R}$ s.t. $AR = I_2$

(5) Consider $A = \begin{bmatrix} \underset{m\times p}{T} & \underset{m\times q}{O} \\ \underset{n\times p}{V} & \underset{n\times q}{W} \end{bmatrix}$ and $\mathcal{R}(V) \subset \mathcal{R}(T)$

Show that $G = \begin{bmatrix} T^- & 0 \\ -W^-VT^- & W^- \end{bmatrix}$ is a g-inverse of A by showing $AGA = A$

(6) Obtain a g-inverse for the matrix $A = \begin{bmatrix} 6 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and

Confirm that $AGA = A$

(Th. 4.9.9.)

(7) $\underset{m\times n}{A}$ $\quad \text{rank}(A) = r \Rightarrow A = \underset{m\times m}{B} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \underset{n\times n}{K}$ $\quad$ ($B, K \equiv$ nonsingular)

Show that if G is of the form $G = K^{-1} \begin{bmatrix} I_r & U \\ V & W \end{bmatrix} B^{-1}$

for any $\underset{r\times(m-r)}{U}, \underset{(n-r)\times r}{V}, \underset{(n-r)\times(m-r)}{W}$ of appropriate dimensions, then $G \equiv$ g-inverse of A

(Th. 4.4.8.)

(8) $\underset{n\times n}{A} \neq 0$ s.t. $A^2 = A$ and $\text{rank}(A) = r \Rightarrow A = \underset{n\times r}{B}\,\underset{r\times n}{L}$ w/ $\text{rank}(B) = \text{rank}(L) = r$

Show that $\text{trace}(A) = \text{rank}(A)$

(9) Consider the matrix $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. a) Compute $P = X(X'X)^{-1}X'$

Show (numerically) that $P^2 = P$

(10) $\underset{n\times n}{A} \equiv$ idempotent, $\underset{n\times n}{B} \equiv$ nonsingular. Show that $B^{-1}AB \equiv$ idempotent

(11) Let $A \equiv m\times n$ show that if $A'A \equiv$ idempotent, then AA' is idempotent.

(12) Let X_0 ~~represent~~ represent a particular solution to $AX = B$ in X then show that if $X^* = X_0 + Z^*$ for some solution Z^* to $AZ = 0$, then $X^* \equiv$ solution to $AX = B$

(13) Show that $\underset{\sim}{X}^* \equiv$ solution to $AX = B$ if $\underset{\sim}{X}^* = A^-B + (I - A^-A)\underset{\sim}{Y}$ for some $\underset{\sim}{Y}$
(By Th. 9.1.2. one solution to $AX = B$ is A^-B)

(14) ~~Show~~ Let $A = \begin{bmatrix} 10 & 5 & 5 \\ 5 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}$ $\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ $\quad \underset{\sim}{b} = \begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix}$

a) obtain two G-inverses of A by i) Inverting $A[1:2, 1:2]$, 0^s elsewhere
ii) " $A[2:3, 2:3]$ " "

$\underset{\sim}{b}$) obtain solutions $\underset{\sim}{X}^*, \underset{\sim}{X}^{**}$ from each of the g-inverses w/ $\underset{\sim}{Y} = 0$ in (13)

(15.) $A \equiv n \times n$ show that $\mathcal{N}(A) = \mathcal{C}(I - A)$ iff $A \equiv$ idempotent

(16). Let $A = \begin{bmatrix} 10 & 5 & 5 \\ 5 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}$ $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix}$

Let $K' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ a) Show that $\mathcal{R}(K') \subset \mathcal{R}(A)$

b) Using your two solutions $\underset{\sim}{x}^*, \underset{\sim}{x}^{**}$ in 14) compute $K'\underset{\sim}{x}^*, K'\underset{\sim}{x}^{**}$

c) Solve for Y^*, a solution to $A'\underset{\sim}{Y} = K$ and

show that $K'\underset{\sim}{x}^* = Y^{*\prime}\underset{\sim}{b}$

(17) Use absorption to solve the following linear system

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 6 & -2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{\sim}{X} = \begin{bmatrix} 58 \\ -27 \\ 70 \\ 63 \end{bmatrix}$$

(18) Let $X_0 = A^- B C^-$ (a particular solution to $AXC = B$)

Show that if $X^* = X_0 + A^-AR(I - CC^-) + (I - A^-A)SCC^- + (I - A^-A)T(I - CC^-)$

for some matrices R, S, T of correct dimensions, then

$X^* \equiv$ a solution to $AXC = B$

(19) Let $Y_1, \ldots, Y_p \equiv$ matrices in linear space $\mathcal{V}$, and $\mathcal{U} \subset \mathcal{V}$.

Let $Z_1, \ldots, Z_p \equiv$ projections of $Y_1, \ldots, Y_p$, respectively on $\mathcal{U}$.

Then for scalars $k_1, \ldots, k_p$, the projection of $k_1 Y_1 + \cdots + k_p Y_p$ on $\mathcal{U}$

is $k_1 Z_1 + \cdots + k_p Z_p$. Prove this result.

(20) Let $\underset{3\times 1}{\underset{\sim}{y}} \in \mathcal{R}^3 \equiv \mathcal{V}$ Let $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathcal{U} \equiv \mathcal{C}(X)$

a) Give 2 points in $\mathcal{U}$

b) Give the projection matrix for $\mathcal{U}$ (it is unique).

20 c) Consider the points $\underset{\sim}{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$ $\underset{\sim}{y}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ $\underset{\sim}{y}_3 = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$

Give their projections $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \underset{\sim}{z}_3$ on $\mathcal{U}$.

d) Show that the projection of $3\underset{\sim}{y}_1 - \underset{\sim}{y}_2 + 2\underset{\sim}{y}_3$ is $3\underset{\sim}{z}_1 - \underset{\sim}{z}_2 + 2\underset{\sim}{z}_3$

e) Show that $(\underset{\sim}{y}_1 - \underset{\sim}{z}_1)'X = 0$

21 Prove Theorems 12.3.4, 12.3.5. (Pages 167–169)

~~22 Let~~ $\cancel{X'X = \begin{bmatrix} 9 & 3 & 3 & 3 \\ 3 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}, \quad \underset{4\times 1}{\underset{\sim}{y}} \in \mathbb{R}^4}$

22 Let $X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $\quad \underset{\sim}{y} \in \mathbb{R}^6 \quad \underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 26 \\ 28 \\ 9 \\ 5 \end{bmatrix}$

a) Obtain $X'X$, $(X'X)^-$, $X'\underset{\sim}{y}$, a solution $\underset{\sim}{b}_0$ to $X'X\underset{\sim}{b} = X'\underset{\sim}{y}$

b) Give the projection of $\underset{\sim}{y}$ onto the $\mathcal{C}(X)$: $\underset{\sim}{z}$

c) show that $(\underset{\sim}{y} - \underset{\sim}{z})'X = 0$

d) Write $\underset{\sim}{y} - \underset{\sim}{z}$ as $A\underset{\sim}{y}$ for some matrix A, and show that $A \in \mathcal{C}^{\perp}(X)$

23 Consider the matrix: $W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.
and $\underset{\sim}{y}, X$ from 22.

a) Show that $\mathcal{C}(W) \subset \mathcal{C}(X)$

b) Give $W'W$, $(W'W)^{-1}$, $W'\underset{\sim}{y}$, solution $\underset{\sim}{b}$ to $W'W\underset{\sim}{b} = W'\underset{\sim}{y}$

$\longrightarrow$

(23) (c) Note that

$$P_X = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Show that $P_X W = W$ numerically and by proof.

(d) Give the projection of $\underset{\sim}{y}$ on $\mathcal{C}^{\perp}(X)$

(24) Consider

$$A = \begin{bmatrix} 6 & 3 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Beginning w/ the elements of the top row, technically how many terms are there in: $\sum (-1)^{\phi_n(i_1,\dots,i_n)} a_{1i_1} a_{2i_2} a_{3i_3} a_{4i_4}$

b) Compute $|A|$

c) Is A nonsingular?

(25) Consider the permutation matrix $P = [\underset{\sim}{u}_5\ \underset{\sim}{u}_3\ \underset{\sim}{u}_2\ \underset{\sim}{u}_6\ \underset{\sim}{u}_1\ \underset{\sim}{u}_4]$

Compute the determinant of P. Hint: compute $\phi_6(5,3,2,6,1,4)$

(26) Let $A = \begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 \\ 1 & -6 \end{bmatrix}$

a) Compute AB

b) Compute $|A|$, $|B|$, $|AB|$

(27) Let $A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 4 & 3 & -2 \\ 6 & 8 & 1 & 0 \end{bmatrix}$ Compute $|A|$

28) Let $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ Compute $|A^3|$ — directly; — Making use of Theorem 13.3.4.

29) Let $A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 1 & 2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a) Compute the minors of a_{11}, a_{12}, a_{13} and their cofactors (α_{1j})

b) Use your results from a) to compute $|A|$

c) Show that $\sum_{j=1}^{3} a_{2j} \alpha_{1j} = 0$

30) Let $A_1 = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$, $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$

Let $B_2 = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$

a) ~~Compute~~ Obtain $\underset{\sim}{x}' A_1 \underset{\sim}{x}$, $\underset{\sim}{x}' B_1 \underset{\sim}{x}$, $\underset{\sim}{x}' B_2 \underset{\sim}{x}$

b) Are these equal for every $\underset{\sim}{x} \in \mathbb{R}^2$?

c) Obtain the following decompositions of A_1:

i) QR ii) LU, iii) Cholesky, iv) Eigenvalue-eigenvector