

Generalized Least Squares – Matrix Estimation – Yule-Walker Method

$$\text{Model: } Y_t = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_t \quad \varepsilon_t + \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \cdots + \rho_q \varepsilon_{t-q} = \eta_t$$

where $\{\varepsilon\} \perp \{\eta\}$. In matrix form:

$$\mathbf{Y} = \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad V\{\boldsymbol{\varepsilon}\} = \boldsymbol{\Gamma}_n$$

1. Obtain Ordinary Least Squares Estimate of $\hat{\boldsymbol{\beta}}_{OLS}$ and Residual Vector (Labelled bols and e in program)

$$\hat{\boldsymbol{\beta}}_{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad (p' \times 1) \quad \mathbf{e} = \mathbf{Y} - \mathbf{X} \hat{\boldsymbol{\beta}}_{OLS} \quad (n \times 1)$$

2. Estimate the autocovariances up to lag q (Labelled gam_q in program)

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} e_t e_{t+h} \quad h = 0, 1, 2, \dots, q$$

3. Obtain the matrix of variances and autocovariances of the residuals to lag q , based on OLS residuals (Labelled gam2_q in program)

$$\hat{\mathbf{\Gamma}}_q = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(q-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \cdots & \hat{\gamma}(q-2) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\gamma}(q-1) & \hat{\gamma}(q-2) & \cdots & \hat{\gamma}(0) \end{bmatrix} \quad (q \times q)$$

4. Obtain a column vector of autocovariances to lag q (Labelled gam1_q in program)

$$\hat{\boldsymbol{\gamma}}_q = \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \\ \vdots \\ \hat{\gamma}(q) \end{bmatrix} \quad (q \times 1)$$

5. Obtain the vector of autoregressive parameter estimates based on the Yule-Walker equations (Labelled rho_q in the program)

$$\hat{\boldsymbol{\rho}} = -\hat{\mathbf{\Gamma}}_q^{-1} \hat{\boldsymbol{\gamma}}_q \quad (q \times 1)$$

6. Obtain the estimate of σ^2 , the variance of the independent random errors η_t (Labelled sigma2_q in the program)

$$\hat{\sigma}^2 = \hat{\gamma}(0) + \hat{\boldsymbol{\rho}}' \hat{\boldsymbol{\gamma}}_q$$

7. Obtain the upper triangular square root (Cholesky Decomposition, using the **root** function in SAS/IML) of $\hat{\mathbf{\Gamma}}_q^{-1}$ (Labelled p_q in program)

$$\hat{\mathbf{\Gamma}}_q^{-1} = \hat{\mathbf{P}}_q' \hat{\mathbf{P}}_q \quad (q \times q)$$

8. Obtain the matrix $\hat{\mathbf{T}} = \hat{\mathbf{V}}^{-1/2}$ (Labelled t_q) in program, it will be lower triangular)

$$\begin{aligned} \mathbf{T}_{11} &= \sqrt{\hat{\sigma}^2} \hat{\mathbf{P}}_{\mathbf{q}} && (q \times q) \\ \mathbf{T}_{12} &= \mathbf{0} && (q \times (n - q)) \\ \mathbf{T}_2 &= \begin{bmatrix} \hat{\rho}_q & \hat{\rho}_{q-1} & \cdots & \hat{\rho}_1 & 1 & 0 & \cdots & 0 \\ 0 & \hat{\rho}_q & \cdots & \hat{\rho}_2 & \cdots & \hat{\rho}_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} && ((n - q) \times n) \\ \hat{\mathbf{T}} &= \begin{bmatrix} \mathbf{T}_{11} & & \\ & \mathbf{T}_2 & \\ & & \end{bmatrix} && (n \times n) \end{aligned}$$

The last array assumes that $n > 2(q + 1)$ which it most certainly should be in practice.

9. Compute the estimates generalized least squares vector $\hat{\beta}_{EGLS}$ (Labelled bgls_q in the program)

$$\hat{\beta}_{EGLS} = (\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{Y} \quad (p' \times 1)$$

10. Estimate the error variance based on the transformed model (Labelled s2gls_q in the program)

$$\hat{\sigma}^2 = [(\mathbf{Y} - \mathbf{X}\hat{\beta}_{EGLS})'\hat{\mathbf{T}}'\hat{\mathbf{T}}(\mathbf{Y} - \mathbf{X}\hat{\beta}_{EGLS})]/(n - p' - q)$$

Note: This denominator differs from the original authors' paper, they use $n - p'$

11. Obtain the (unscaled) variance-covariance matrix of $\hat{\beta}_{EGLS}$ (Labelled c_q in the program)

$$(\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{X})^{-1} = (\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1} \quad (p' \times p')$$

12. Obtain the estimated standard errors of $\hat{\beta}_{EGLS}$ (Labelled sbgls_q in the program)

- Multiply the the previous matrix by $\hat{\sigma}^2$
- Obtain the square root of the diagonal elements

13. Obtain residual mean squares for estimates of autoregressive parameters ρ (Labelled s2rho_q in program)

$$s^2 = [\hat{\gamma}(0) + \hat{\rho}'\hat{\gamma}_q]/(n - p' - q)$$

Note: This denominator differs from the original authors' paper, they use $n - q$

14. Obtain the estimated standard error of ρ_i (Labelled serho_qi)

$$s\{\hat{\rho}_i\} = \sqrt{s^2\hat{\gamma}^{ii}} \quad i = 1, \dots, q$$

where $\hat{\gamma}^{ii}$ is the i^{th} diagonal element of $(\hat{\mathbf{T}}_q)^{-1}$

15. Obtain the t -statistics for ρ_i $i = 1, \dots, q$ (Labelled trho_qi and strung together into vectors trho_q)

$$t = \frac{\hat{\rho}_i}{s\{\hat{\rho}_i\}} \quad i = 1, \dots, q$$

16. These are compared with critical values of the t -distribution with $n - p' - q$