Chapter 11 - Factorial TRT Designs

2^n Factorial Experiments for Evaluating Several TRTs.

- Labelling treatments' levels - 0 for "Low," 1 for "High"
- Uppercase letters ≡ Factors (e.g. A, B, C)
- Lowercase letters used for TRT labels (letter appears if factor is @ high level).

Example - 2^3 Factorial Experiment

3 FACTORS - EACH @ 2 LEVELS

EXPERIMENT TO MEASURE EFFECTS OF 3 NUTRIENTS ON PLANT GROWTH - NUTRIENTS ARE FACTORS (PRESENT/ABSENT ≡ LEVELS)

Description	TRT	A	B	C	I	A	B	C	AB	AC	BC	ABC	y
					Defining Contrasts								
A LO/B LO/C LO	(1)	0	0	0	+	−	−	−	+	+	+	−	y_{000}
A HI/B LO/C LO	a	1	0	0	+	+	−	−	−	−	+	+	y_{100}
A LO/B HI/C LO	b	0	1	0	+	−	+	−	−	+	−	+	y_{010}
A LO/B LO/C HI	c	0	0	1	+	−	−	+	+	−	−	+	y_{001}
A HI/B HI/C LO	ab	1	1	0	+	+	+	−	+	−	−	−	y_{110}
A HI/B LO/C HI	ac	1	0	1	+	+	−	+	−	+	−	−	y_{101}
A LO/B HI/C HI	bc	0	1	1	+	−	+	+	−	−	+	−	y_{011}
A HI/B HI/C HI	abc	1	1	1	+	+	+	+	+	+	+	+	y_{111}
Divisor (sum of +'s in column)					8	4	4	4	4	4	4	4	

To get effects for each MAIN Effect & Interaction, multiply each observation by the +/- in the column for that effect, sum and divide by column divisor.

$$I = \frac{1}{8}\left[y_{000} + y_{100} + y_{010} + y_{001} + y_{110} + y_{101} + y_{011} + y_{111}\right]$$

$$A = \frac{1}{4}\left[(y_{100} + y_{110} + y_{101} + y_{111}) - (y_{000} - y_{010} - y_{001} - y_{011})\right]$$

$$B = \frac{1}{4}\left[(y_{010} + y_{110} + y_{011} + y_{111}) - (y_{000} + y_{100} + y_{001} + y_{101})\right]$$

$$C = \frac{1}{4}\left[(y_{001} + y_{101} + y_{011} + y_{111}) - (y_{000} + y_{100} + y_{010} + y_{110})\right]$$

$$AB = \frac{1}{4}\left[\underbrace{(y_{000} + y_{001} + y_{110} + y_{111})}_{\substack{\text{A \& B appear "together"} \\ \text{@ Low or High Levels}}} - \underbrace{(y_{100} + y_{010} + y_{101} + y_{011})}_{\substack{\text{A @ Low when B @ High} \\ \text{\& vice versa}}}\right]$$

$$AC = \frac{1}{4}\left[(y_{000} + y_{010} + y_{101} + y_{111}) - (y_{100} + y_{001} + y_{110} + y_{011})\right]$$

$$BC = \frac{1}{4}\left[(y_{000} + y_{100} + y_{011} + y_{111}) - (y_{010} + y_{001} + y_{110} + y_{101})\right]$$

$$ABC = \frac{1}{4}\left[(y_{100} + y_{010} + y_{001} + y_{111}) - (y_{000} + y_{110} + y_{101} + y_{011})\right]$$

Recall NOTION OF INTERACTION: Effect of A depends on level of B => AB INTERACTION EXISTS.

$$(A|B=0) = \frac{1}{2}\left[(y_{100} + y_{101}) - (y_{000} + y_{001})\right]$$

$$(A|B=1) = \frac{1}{2}\left[(y_{110} + y_{111}) - (y_{010} + y_{011})\right]$$

$$AB = \frac{1}{2}\left[(A|B=1) - (A|B=0)\right]$$

$$(AB|C=0) = \frac{1}{2}\left[(y_{000} + y_{110}) - (y_{100} + y_{010})\right]$$

$$(AB|C=1) = \frac{1}{2}\left[(y_{001} + y_{111}) - (y_{101} + y_{011})\right]$$

$$ABC = \frac{1}{2}\left[(AB|C=1) - (AB|C=0)\right]$$

NOTE THESE Y-values are means of r replicates when conducted in CRD (or block design)

Standard error of Effect Estimate (Based on CRD) – General Case 2^n Factorial

$\bar{Y} \equiv$ mean of r replicates for each combination of factor levels $\Rightarrow$ $\sigma^2_{\bar{Y}} = \frac{\sigma^2}{r}$ ($\sigma^2 \equiv$ experimental error variance)

Independent across treatments.

$S^2_{AB\ldots} = V\left(\frac{1}{2^{n-1}} l_{AB\ldots}\right)$ where $l_{AB\ldots} \equiv$ Linear contrast of TRT means (Effect Estimate) (NOT USING Divisor)

$l_{AB\ldots} = \sum_i k_i \bar{Y}_{i.}$ $\quad k_i \equiv +/-1$

$AB\ldots = \frac{1}{2^{n-1}} l_{AB\ldots}$

$$V(l_{AB\ldots}) = V\left(\sum_i k_i \bar{Y}_{i.}\right) = \sum_i k_i^2 V(\bar{Y}_{i.})$$

$$= \frac{\sigma^2}{r} \sum_i (\pm 1)^2 = \frac{\sigma^2}{r}(2^n)$$

$$\Rightarrow S^2_{AB\ldots} = V\left(\frac{1}{2^{n-1}} l_{AB\ldots}\right) = \left[\frac{1}{2^{n-1}}\right]^2 V(l_{AB\ldots})$$

$$= \frac{1}{2^{2n-2}}\left(\frac{\sigma^2}{r} 2^n\right) = \frac{\sigma^2}{r}\left(\frac{2^n}{2^{2n} 2^{-2}}\right) = \frac{4\sigma^2}{r 2^n}$$

$$\Rightarrow \boxed{S_{AB\ldots} = \sqrt{\frac{4\,MSE}{r 2^n}}}$$

Recall Sum of Squares for Contrast (Ch. 3)

$$\underbrace{SSC = \frac{(\sum k_i \bar{y}_{i\cdot})^2}{\sum (k_i^2/r_i)} = r\frac{(\sum k_i \bar{y}_{i\cdot})^2}{\sum k_i^2}}$$ (when r_i are all =)

Equations 3.6, 3.7 (P. 77)

$$SSAB\ldots = r\frac{(\sum k_i \bar{y}_{i\cdot})^2}{\sum k_i^2}$$

where $\sum k_i \bar{y}_{i\cdot} = l_{AB\ldots}$

$$\sum k_i^2 = \sum_{i=1}^{2^n} (\pm 1)^2 = 2^n$$

$$\Rightarrow SSAB\ldots = r\frac{(l_{AB\ldots})^2}{2^n}$$

EXAMPLE - COCHRAN & COX (1957) - P. 158

4 Factors (Manure, NITROGEN, PHOSPHOROUS, POTASSIUM)
(M) (N) (P) (K)

Conducted in 4 Replicates

Rep	MEAN	Rep MEAN $-\bar{y}$	$16(\overline{REP}-\bar{y})^2$
1	58.4375	3.84375	236.390625
2	53.0000	-1.59375	40.640625
3	55.8125	1.21875	23.765625
4	51.1250	-3.46875	192.515625
ALL	54.59375	0	493.3125

TRT	Rep 1	2	3	4	$\bar{y}$
(1)	32	43	27	19	30.25
m	47	41	48	45	45.25
n	26	36	24	18	26.00
p	29	39	27	28	30.75
k	35	42	56	35	42.00
mn	61	76	56	64	64.25
mp	51	34	40	48	43.25
mk	63	41	60	53	54.25
np	36	31	32	30	32.25
nk	80	68	75	67	72.50
pk	40	44	53	36	43.25
mnp	76	65	70	63	68.50
mnk	100	68	87	66	80.25
mpk	64	39	75	72	62.50
npk	105	99	74	73	87.75
mnpk	90	82	89	101	90.50

SS TOTAL =

$\sum\sum(y_{ij\cdot\cdot}-\bar{y})^2$

$=(32-54.59375)^2$

$+\dots+(101-54.59375)^2$

$=31359.4$

DEFINING CONTRASTS

TRT	I	M	N	P	K	MN	MP	MK	NP	NK	PK	MNP	MNK	MPK	NPK	MNPK
(1)	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+
m	+	+	−	−	−	−	−	−	+	+	+	+	+	+	−	−
n	+	−	+	−	−	−	+	+	−	−	+	+	+	−	+	−
p	+	−	−	+	−	+	−	+	−	+	−	+	−	+	+	−
k	+	−	−	−	+	+	+	−	+	−	−	−	+	+	+	−
mn	+	+	+	−	−	+	−	−	−	−	+	−	−	+	+	+
mp	+	+	−	+	−	−	+	−	−	+	−	−	+	−	+	+
mk	+	+	−	−	+	−	−	+	+	−	−	+	−	−	+	+
np	+	−	+	+	−	−	−	+	+	−	−	+	+	+	−	+
nk	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
pk	+	−	−	+	+	+	−	−	−	−	+	+	+	−	−	+
mnp	+	+	+	+	−	+	+	−	+	−	−	+	−	−	−	−
mnk	+	+	+	−	+	+	−	+	−	+	−	−	+	−	−	−
mpk	+	+	−	+	+	−	+	+	−	−	+	−	−	+	−	−
npk	+	−	+	+	+	−	−	−	+	+	+	−	−	−	+	−
mnpk	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Main Effect for Factor M (MANURE)

$$l_M = \Big[(45.25 + 64.25 + 43.25 + 54.25 + 68.50 + 80.25 + 62.50 + 90.50)$$

$$- (30.25 + 26.00 + 30.75 + 42.00 + 32.25 + 72.50 + 43.25 + 87.75) \Big]$$

$$= [\, 508.75 - 364.75 \,] = 144.00$$

$$\Rightarrow \boxed{M = \frac{l_M}{2^{n-1}} = \frac{144}{2^{(4-1)}} = \frac{144}{8} = 18.0 \qquad SSM = 4\,\frac{(144)^2}{16} = 5184 \qquad MSM = \frac{SSM}{1} = 5184}$$

$\boxed{MSE = 90.5 \text{ (see c \& c (p. 160))} \quad df_E = 45}$

H_0: NO MANURE EFFECT (POPULATION $M = 0$)

H_A: MANURE EFFECT (Population $M \neq 0$)

t-test T.S.

$$t_0 = \frac{M}{s_M} = \frac{18.0}{\sqrt{\frac{4\,MSE}{4(2^4)}}} = \frac{18.0}{\sqrt{\frac{4(90.5)}{4(16)}}} = \frac{18.0}{\sqrt{\frac{90.5}{16}}}$$

$$= \frac{18.0}{2.38} = 7.56 \qquad RR: |t_0| \geq t_{.025,45} \approx 2.015$$

F-test T.S.

$$F_0 = \frac{MSM}{MSE} = \frac{5184}{90.5} = 57.28 = (7.56)^2$$

$$RR: F_0 \geq F_{.05,1,45} = 4.06$$

WOULD REPEAT THIS PROCESS FOR EACH MAIN EFFECT & INTERACTION

2^n Factorial Experiments in Incomplete Blocks

- Good – Increase Precision (Homogeneous units within Blocks)
- BAD – TREATMENT EFFECTS CONFOUNDED WITH (INDISTINGUISHABLE FROM) BLOCK EFFECTS

EXAMPLE: 2^3 Factorial in 2 Blocks of Size 4

$$l_A = (a + ab + ac + abc) - ((1) + b + c + bc)$$

A would be confounded w/ blocks if first 4 measurements obtained from block 1, last 4 measurements obtained from block 2

i.e.

Block 1	Block 2
a	(1)
ab	b
ac	c
abc	bc

A effect is confounded w/ block effect

GOAL: WANT TO CONFOUND HIGHER ORDER INTERACTIONS w/ BLOCKS (NOT MAIN EFFECTS OR LOWER ORDER INTERACTIONS)

$$l_{ABC} = \left[(a + b + c + abc) - ((1) + ab + ac + bc)\right]$$

Place these 2 GROUPS IN BLOCKS 1 & 2, respectively

Block 1	Block 2
a	(1)
b	ab
c	ac
abc	bc

NOTE That each main effect & 2-factor interaction have 2 positive and 2 negative terms from each block.

Partial Confounding

Confound Different interaction in different replicates (sets of incomplete blocks)

EXAMPLE – 2^3 in 3 Replicates, Each w/ 2 blocks of size 4.

Confound AB in Rep 1, AC in Rep 2, BC in Rep 3

$$AB = ((1) + c + ab + abc) - (a + b + ac + bc)$$
$$AC = ((1) + b + ac + abc) - (a + c + ab + bc)$$
$$BC = ((1) + a + bc + abc) - (b + c + ab + ac)$$

Rep 1 (AB)		Rep 2 (AC)		Rep 3 (BC)	
(1)	a	(1)	a	(1)	b
c	b	b	c	a	c
ab	ac	ac	ab	bc	ab
abc	bc	abc	bc	abc	ac
Blk 1	Blk 2	Blk 3	Blk 4	Blk 5	Blk 6

l_A, l_B, l_C, l_{ABC} computed exactly as before

To compute l_{AB}, l_{AC}, l_{BC}:

1. Work w/ TOTALS, not MEANS
2. Obtain $\sum_i k_i Y_{i.}$ for each 2-FI
3. Subtract off difference in block means in confounded blocks

EXAMPLE (KUEHL, EX. 11.2, P. 373-375)

TRT	(TOTAL) $y_{i.}$	(mean) $\bar{y}_{i.}$	A	B	C	ABC	AB	AC	BC
			Defining Contrasts (ME ! 3FI)				2-FI		
(1)	78	26	−	−	−	−	+	+	+
a	105	35	+	−	−	+	−	−	+
b	96	32	−	+	−	+	−	+	−
c	96	32	−	−	+	+	+	−	−
ab	141	47	+	+	−	−	+	−	−
ac	120	40	+	−	+	−	−	+	−
bc	108	36	−	+	+	−	−	−	+
abc	132	44	+	+	+	+	+	+	+

MEANS:

$\ell_A = (35+47+40+44) - (26+32+32+36) = 166 - 126 = 40$

$\ell_B = (32+47+36+44) - (26+35+32+40) = 159 - 133 = 26$

$\ell_C = (32+40+36+44) - (26+35+32+47) = 152 - 140 = 12$

$\ell_{ABC} = (35+32+32+44) - (26+47+40+36) = 143 - 149 = -6$

BLOCK Effects

BC: $\text{BLOCK}_1 - \text{BLOCK}_2 = 126 - 146 = -20$ (20 higher when BC has − term)

AC: $\text{BLOCK}_3 - \text{BLOCK}_4 = 135 - 155 = -20$ (20 higher when AC has − term)

AB: BLOCK 5 − BLOCK 6 = 161 − 153 = 8 (8 higher when AB has + term)

TOTALS

AB: $\sum_i k_i y_{i.} = (78+96+141+132) - (105+96+120+108) = 447 - 429 = 18$

AC: $\sum k_i y_{i.} = (78+96+120+132) - (105+96+141+108) = 426 - 450 = -24$

BC: $\sum k_i y_{i.} = (78+105+108+132) - (96+96+141+120) = 423 - 453 = -30$

overall — Removing confounded block

$$l_{AB} = \frac{\sum k_i y_i - (\text{Block}_5 - \text{Block}_6)}{3-1} = \frac{18-8}{2} = 5$$

$$l_{AC} = \frac{-24-(-20)}{2} = -2$$

$$l_{BC} = \frac{-30-(-20)}{2} = -5$$

$$SSA = \frac{r l_A^2}{2^n} = \frac{3(40)^2}{2^3} = 600 = MSA$$

$$SSB = \frac{r l_B^2}{2^n} = \frac{3(26)^2}{8} = 253.5 = MSB$$

$$SSC = \frac{3(12)^2}{8} = 54 = MSC$$

$$SSABC = \frac{3(-6)^2}{8} = 13.5 = MSABC$$

$$SSAB = \frac{(3-1)(5)^2}{8} = 6.25 = MSAB$$

$$SSAC = \frac{(3-1)(-2)^2}{8} = 1.00 = MSAC$$

$$SSBC = \frac{(3-1)(-5)^2}{8} = 6.25 = MSBC$$

Rep Means: $\overline{Rep_1} = 34.00$ $\overline{Rep_2} = 36.25$ $\overline{Rep_3} = 39.25$ $\bar{Y}_{\cdots} = 36.50$

$$SS(\text{Reps}) = 8\left[(34-36.5)^2 + (36.25-36.50)^2 + (39.25-36.50)^2\right]$$

$$= 8[6.25 + .0625 + 7.5625] = 111 \Rightarrow MS\text{Reps} = 55.5$$

$$SS[BL(\text{Reps})] = 4\left[(31.5-34)^2 + (36.50-34)^2 + (33.75-36.25)^2 + (38.75-36.25)^2 + (40.25-39.25)^2 + (38.25-39.25)^2\right]$$

$$= 4[6.25 + 6.25 + 6.25 + 6.25 + 1 + 1] = 108 \qquad MS(BL(\text{Rep})) = \frac{108}{3} = 36$$

ANOVA

$SS_{TOTAL} = \sum\sum (y - 36.5)^2 = 1316.00$

SOURCE	df	SS	MS	F_0
Replicates	3-1=2	111	55.5	—
Blocks(Reps)	3(2-1)=3	108	36	—
A	2-1=1	600	600	40.6 *
B	2-1=1	253.5	253.5	17.2 *
C	2-1=1	54	54	3.7
ABC	(1)(1)(1)=1	13.5	13.5	0.9
AB (I/II)	(1)(1)=1	6.25	6.25	0.4
AC (I/III)	(1)(1)=1	1.00	1.00	0.1
BC(II/III)	(1)(1)=1	6.25	6.25	0.4
ERROR	23-2-3-7(1)=11	162.50	14.77	—
TOTAL	24-1=23	1316.0		

$F_{.05,1,11} = 4.84$