Chapter 7

Random Effects Model for Factorial TRT DESIGNS

Model: $y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk}$ $\quad i=1,\dots,a$; $j=1,\dots,b$; $k=1,\dots,r$

$$\left.\begin{array}{l} a_i \sim NID(0,\sigma_a^2) \\ b_j \sim NID(0,\sigma_b^2) \\ (ab)_{ij} \sim NID(0,\sigma_{ab}^2) \\ e_{ijk} \sim NID(0,\sigma^2) \end{array}\right\} \text{Independent}$$

$$Cov(y_{ijk}, y_{i'j'k'}) = Cov(\mu + a_i + b_j + (ab)_{ij} + e_{ijk},\ \mu + a_{i'} + b_{j'} + (ab)_{i'j'} + e_{i'j'k'})$$

$i=i', j=j', k=k' \Rightarrow Cov(y_{ijk}, y_{i'j'k'}) = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$

$i=i', j=j'\ k\neq k' \Rightarrow Cov(y_{ijk}, y_{i'j'k'}) = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2$

$i=i', j\neq j' \Rightarrow Cov(y_{ijk}, y_{i'j'k'}) = \sigma_a^2$

$i\neq i', j=j' \Rightarrow Cov(y_{ijk}, y_{i'j'k'}) = \sigma_b^2$

$i\neq i', j\neq j' = Cov(y_{ijk}, y_{i'j'k'}) = 0$

$E[y_{ijk}] = \mu \quad \forall\ i,j,k$

$$SSA = \sum_i\sum_j\sum_k (\bar{y}_{i..} - \bar{y}_{...})^2 = br \sum_i \bar{y}_{i..}^2 - abr\,\bar{y}_{...}^2$$

$$SSB = \sum_i\sum_j\sum_k (\bar{y}_{.j.} - \bar{y}_{...})^2 = ar \sum_j \bar{y}_{.j.}^2 - abr\,\bar{y}_{...}^2$$

$$SSAB = \sum_i\sum_j\sum_k (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$

(see chapter 6 course notes)

$$= r\sum_i\sum_j \bar{y}_{ij.}^2 - br\sum_i \bar{y}_{i..}^2 - ar\sum_j \bar{y}_{.j.}^2 + abr\,\bar{y}_{...}^2$$

$$SSE = \sum_i\sum_j\sum_k (y_{ijk} - \bar{y}_{ij.})^2 = \sum_i\sum_j\sum_k y_{ijk}^2 - r\sum_i\sum_j \bar{y}_{ij.}^2$$

$$E[y_{ijk}] = \mu \qquad V[y_{ijk}] = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$$

$$\Rightarrow \boxed{E[y_{ijk}^2] = \mu^2 + \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2}$$

$$E[\bar{y}_{ij.}] = \mu$$

$$V[\bar{y}_{ij.}] = V\left[\frac{1}{r}\sum_k y_{ijk}\right] = \frac{1}{r^2} V\left[\sum_k y_{ijk}^2\right]$$

$$= \frac{1}{r^2}\left\{ \sum_{k=1}^{r} V(y_{ijk}) + 2\sum\sum_{k<k'} Cov(y_{ijk}, y_{ijk'}) \right\}$$

$$= \frac{1}{r^2}\left\{ r(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2) + 2\binom{r}{2}(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2) \right\}$$

$$= \frac{1}{r^2}\left\{ r(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2) + r(r-1)(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2) \right]$$

$$\Rightarrow \boxed{E[\bar{y}_{ij.}^2] = \mu^2 + \frac{1}{r^2}\left\{ r(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2) + r(r-1)(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2) \right\}}$$

$$E[\bar{y}_{i\cdot\cdot}] = \mu$$

$$V[\bar{y}_{i\cdot\cdot}] = V\left[\frac{1}{br}\sum_j \sum_k y_{ijk}\right]$$

br term

$$= \left(\frac{1}{br}\right)^2 V\left[\sum_j\sum_k y_{ijk}\right] = \left(\frac{1}{br}\right)^2 \left[\sum_j\sum_k V(y_{ijk})\right.$$

$$+ 2\sum_j \underbrace{\sum\sum_{k<k'} Cov(y_{ijk}, y_{ijk'})}_{br(r-1)\text{ terms}} + \underbrace{2\sum\sum_{j<j'} \sum_k\sum_{k'} Cov(y_{ijk}, y_{ij'k'})}_{b(b-1)r^2\text{ terms}}\left.\right]$$

$br + br(r-1) + b(b-1)r^2 = b^2r^2$ ✓

$$= \left(\frac{1}{br}\right)^2\left[br(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma_e^2) + br(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2)\right.$$

$$\left. + b(b-1)r^2\sigma_a^2\right]$$

$$\Rightarrow \boxed{E[\bar{y}_{i\cdot\cdot}^2] = \mu^2 + \left(\frac{1}{br}\right)^2\left[br(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma_e^2) + br(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + b(b-1)r^2\sigma_a^2\right]}$$

By direct analogy (since labels of A & B could have been reversed)

$$\boxed{E[\bar{y}_{\cdot j\cdot}^2] = \mu^2 + \left(\frac{1}{ar}\right)^2\left[ar(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma_e^2) + ar(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + a(a-1)r^2\sigma_b^2\right]}$$

$$E[\bar{Y}_{\cdots}] = \mu$$

$$V[\bar{Y}_{\cdots}] = V\left[\frac{1}{abr}\sum_i\sum_j\sum_k Y_{ijk}\right] = \left(\frac{1}{abr}\right)^2 V\left[\sum_i\sum_j\sum_k Y_{ijk}\right]$$

$$= \left(\frac{1}{abr}\right)^2 \Big\{ \underbrace{\sum_i\sum_j\sum_k V(Y_{ijk})}_{abr\text{ terms}} + \underbrace{2\sum_i\sum_j\sum\sum_{k<k'} Cov(Y_{ijk}, Y_{ijk'})}_{ab(r)(r-1)\text{ terms}}$$

$$+ \underbrace{2\sum_i\sum\sum_{j<j'}\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{ij'k'})}_{ab(b-1)r^2\text{ terms}} + \underbrace{2\sum\sum_{i<i'}\sum_j\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'jk'})}_{a(a-1)br^2\text{ terms}}$$

$$+ \underbrace{2(2)\sum\sum_{i<i'}\sum\sum_{j<j'}\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'j'k'})}_{a(a-1)b(b-1)r^2\text{ terms}} \qquad (*)$$

Check: $abr + abr(r-1) = abr^2$

$ab(b-1)r^2 = ab^2r^2 - abr^2$

ab^2r^2

$a(a-1)br^2 = a^2br^2 - abr^2$

$a(a-1)b(b-1)r^2 = a^2b^2r^2 - a^2br^2 - ab^2r^2 + abr^2$

$= a^2b^2r^2 \cancel{-a^2br^2} \cancel{-ab^2r^2} \cancel{+abr^2} \cancel{+a^2br^2} \cancel{-abr^2} \cancel{+ab^2r^2}$ ✓

$$\Rightarrow (*) = \left(\frac{1}{abr}\right)^2 \Big\{ abr(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_e^2) + abr(r-1)(\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2)$$

$$+ ab(b-1)r^2\,\sigma_a^2 + a(a-1)br^2\,\sigma_b^2 + a(a-1)b(b-1)r^2(0) \Big\}$$

$$\Rightarrow E[\bar{Y}_{\cdots}^2] = \mu^2 + \underline{\qquad\qquad\qquad\qquad}$$

$$E[SSE] = E\left[\sum_i\sum_j\sum_k y_{ijk}^2 - \sum_i\sum_j \bar{y}_{ij.}^2\right]$$

$$= abr\,E[y_{ijk}^2] - abr\,E[\bar{y}_{ij.}^2]$$

$$= abr\left\{[\mu^2 + \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2] - \left[\mu^2 + \frac{1}{r^2}\left[(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2)r + r(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2)\right]\right]\right\}$$

$$= abr\left\{\mu^2(1-1) + \sigma_a^2\left[1 - \frac{r}{r^2} - \frac{r(r-1)}{r^2}\right] + \sigma_b^2\left[1 - \frac{r}{r^2} - \frac{r(r-1)}{r^2}\right] + \sigma_{ab}^2\left[1 - \frac{r}{r^2} - \frac{r(r-1)}{r^2}\right] + \sigma^2\left[1 - \frac{r}{r^2}\right]\right\}$$

$$= abr\left\{\mu^2(0) + \sigma_a^2\left[\frac{r^2 - r - \cancel{r^2} + r}{\cancel{r^2}}\right]^{0} + 0\sigma_b^2 + 0\sigma_{ab}^2 + \frac{r^2 - r}{r^2}\sigma^2\right\}$$

$$= abr\left[\frac{r(r-1)}{r^2}\right]\sigma^2 = ab(r-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSE] = E\left[\frac{SSE}{ab(r-1)}\right] = \sigma^2}$$

$$E[MSAB] = E\left[r\sum_i\sum_j \bar{y}_{ij.}^2 - br\sum_i \bar{y}_{i..}^2 - ar\sum_j \bar{y}_{.j.}^2 + abr\bar{y}_{...}^2\right]$$

$$\cancel{= abr\left[\mu^2 + \frac{1}{r^2}\left[r(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2) + r(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2)\right]\right]}$$

$$-\frac{abr}{(br)^2}\left[\mu^2\right.$$

$$= abr\left[\mu^2 + \frac{1}{r^2}\left[r(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2)+r(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2)\right]\right]$$

$$- abr\left[\mu^2 + \left(\frac{1}{br}\right)^2\left[br(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2)+br(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + b(b-1)r^2\sigma_a^2\right]\right]$$

$$-abr\left[\mu^2 + \left(\frac{1}{ar}\right)^2\left[ar(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2)+ar(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + a(a-1)r^2\sigma_b^2\right]\right]$$

$$+abr\left[\mu^2 + \left(\frac{1}{abr}\right)^2\left[abr(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma_e^2)+abr(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + ab(b-1)r^2\sigma_a^2 + a(a-1)br^2\sigma_b^2\right]\right]$$

Putting coefficients of terms together

$$\mu^2: \quad abr - abr - abr + abr = 0$$

$$\sigma_a^2: \quad \frac{abr^2}{r^2} + \frac{ab^2(r-1)}{r^2} - \frac{ab^2r^2}{(br)^2} - \frac{ab^2r^2(r-1)}{(br)^2} - \frac{ab^2r^3(b-1)}{(br)^2}$$

$$- \frac{a^2br^2}{(ar)^2} - \frac{a^2br^2(r-1)}{(ar)^2} + \frac{(abr)^2}{(abr)^2} + \frac{(abr)^2(r-1)}{(abr)^2}$$

$$+ \frac{(abr)^2(b-1)r}{(abr)^2}$$

$$= ab + ab - \frac{ab}{r} \overbrace{\overbrace{-a - a(r-1)}^{ar} - ar(b-1)}^{arb} \overbrace{-b - b(r-1)}^{br}$$

$$+1 + (r-1) + (b-1)r = a$$

$$= ab + ab(r-1) - a - a(r-1) - ar(b-1)$$

$$- b - b(r-1) + 1 + r - 1 + (b-1)r$$

$$= abr - ar - ar(b-1) - br + r$$

$$= abr - abr - br + rb = 0$$

Similarly, coefficient for $\sigma_b^2 = 0$ —

$$\sigma_{ab}^2: \quad \frac{abr^2}{r^2} + \frac{abr^2(r-1)}{r^2} - \frac{ab^2r^2}{(br)^2} - \frac{ab^2r^2(r-1)}{(br)^2}$$

$$- \frac{a^2br^2}{(ar)^2} - \frac{a^2br^2(r-1)}{(ar)^2} + \frac{(abr)^2}{(abr)^2} + \frac{(abr)^2(r-1)}{(abr)^2}$$

$$= ab + ab(r-1) - a - a(r-1) - b - b(r-1) + 1 + (r-1)$$

$$= abr - ar - br + r =$$

$$= r(a-1)(b-1)$$

$$\sigma_e^2: \quad \frac{abr^2}{r^2} - \frac{ab^2r^2}{(br)^2} - \frac{a^2br^2}{(ar)^2} + \frac{(abr)^2}{(abr)^2}$$

$$= ab - a - b + 1 = (a-1)(b-1)$$

$$\Rightarrow E[SSAB] = r(a-1)(b-1)\sigma_{ab}^2 + (a-1)(b-1)\sigma_e^2$$

$$\Rightarrow E[MSAB] = E\left[\frac{SSAB}{(a-1)(b-1)}\right] = r\sigma_{ab}^2 + \sigma_e^2$$

$$E[SSA] = E\left[br\sum_i \bar{y}_{i..}^2 - abr\,\bar{y}_{...}^2\right]$$

$$= abr\left[\mu^2 + \left(\frac{1}{br}\right)^2\left[br(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma^2) + br(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + b(b-1)r^2\sigma_a^2\right]\right]$$

$$- abr\left[\mu^2 + \left(\frac{1}{abr}\right)^2\left[abr(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2+\sigma_e^2) + abr(r-1)(\sigma_a^2+\sigma_b^2+\sigma_{ab}^2) + ab(b-1)r^2\sigma_a^2 + a(a-1)br^2\sigma_b^2\right]\right]$$

Coefficients of

μ^2: $abr - abr = 0$

$$\sigma_a^2: \quad \frac{ab^2r^2}{(br)^2} + \frac{ab^2r^2(r-1)}{(br)^2} + \frac{ab^2(b-1)r^3}{(br)^2} - \frac{(abr)^2}{(abr)^2} - \frac{(abr)^2(r-1)}{(abr)^2} - \frac{a^2b^2(b-1)r^3}{(abr)^2}$$

$$= \underbrace{a + a(r-1) + a(b-1)r}_{=\,abr} - \underbrace{1 - (r-1) - (b-1)r}_{br} = br(a-1)$$

$$\sigma_B^2: \quad \frac{ab^2r^2}{(br)^2} + \frac{ab^2r^2(r-1)}{(br)^2} - \frac{(abr)^2}{(abr)^2} - \frac{(abr)^2(r-1)}{(abr)^2} - \frac{a^2(a-1)b^2r^3}{(abr)^2}$$

$$= a + a(r-1) - 1 - (r-1) - (a-1)r$$

$$= ar - ar = 0$$

$$\sigma_{ab}^2: \quad \frac{ab^2r^2}{(br)^2} + \frac{ab^2r^2(r-1)}{(br)^2} - \frac{(abr)^2}{(abr)^2} - \frac{(abr)^2(r-1)}{(abr)^2}$$

$$= a + a(r-1) - 1 - (r-1)$$

$$= ar - r = r(a-1)$$

$$\sigma_e^2: \quad \frac{ab^2r^2}{(br)^2} - \frac{(abr)^2}{(abr)^2} = a - 1$$

$$\Rightarrow E[SSA] = br(a-1)\sigma_a^2 + r(a-1)\sigma_{ab}^2 + (a-1)\sigma_e^2$$

$$\Rightarrow E[MSA] = E\left[\frac{SSA}{a-1}\right] = br\sigma_a^2 + r\sigma_{ab}^2 + \sigma_e^2$$

By analogy:

$$E[MSB] = ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma_e^2$$

ANOVA - 2 Factor Random Effects Model

Source	df	SS	MS	E[MS]
A	$a-1$	$br\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2$	$\frac{SSA}{a-1}$	$\sigma_e^2 + r\sigma_{ab}^2 + br\sigma_a^2$
B	$b-1$	$ar\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2$	$\frac{SSB}{b-1}$	$\sigma_e^2 + r\sigma_{ab}^2 + ar\sigma_b^2$
A*B	$(a-1)(b-1)$	$r\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	$\frac{SSAB}{(a-1)(b-1)}$	$\sigma_e^2 + r\sigma_{ab}^2$
Error	$ab(r-1)$	$\sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{ij.})^2$	$\frac{SSE}{ab(r-1)}$	σ_e^2
Total	$abr-1$	$\sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{...})^2$	—	

Test For Interaction

$H_0: \sigma_{ab}^2 = 0 \quad H_A: \sigma_{ab}^2 > 0$

T.S. $F_{AB} = \frac{MSAB}{MSE}$

RR: $F_{AB} \geq F_{\alpha,(a-1)(b-1),\,ab(r-1)}$

P-value $= P(F \geq F_{AB} \mid F \sim F_{(a-1)(b-1),\,ab(r-1)})$

Tests For Main Effects

$H_0: \sigma_A^2 = 0 \quad H_A: \sigma_A^2 > 0$

T.S. $F_A = \frac{MSA}{MSAB}$

RR: $F_A \geq F_{\alpha,\,a-1,\,(a-1)(b-1)}$

P-value $= P(F \geq F_A \mid F \sim F_{a-1,\,(a-1)(b-1)})$

$H_0: \sigma_B^2 = 0 \quad H_A: \sigma_B^2 > 0$

T.S. $F_B = \frac{MSB}{MSAB}$

RR: $F_B \geq F_{\alpha,\,b-1,\,(a-1)(b-1)}$

P-val $= P(F \geq F_B \mid F \sim F_{b-1,\,(a-1)(b-1)})$

3-FACTOR MODEL

$$Y_{ijkl} = \mu + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + e_{ijkl}$$

$$i = 1, \dots, a \quad j = 1, \dots, b \quad k = 1, \dots, c \quad l = 1, \dots, r$$

$a_i \sim NID(0, \sigma_a^2)$ $\quad b_j \sim NID(0, \sigma_b^2)$ $\quad c_k \sim NID(0, \sigma_c^2)$

$(ab)_{ij} \sim NID(0, \sigma_{ab}^2)$ $\quad (ac)_{ik} \sim NID(0, \sigma_{ac}^2)$ $\quad (bc)_{jk} \sim NID(0, \sigma_{bc}^2)$

$(abc)_{ijk} \sim NID(0, \sigma_{abc}^2)$ $\quad e_{ijkl} \sim NID(0, \sigma^2)$

All terms are pairwise mutually independent

$$SSA = bcr \sum_i (\bar{Y}_{i\dots} - \bar{Y}_{\dots\dots})^2 \qquad E[MSA] = \sigma^2 + r\sigma_{abc}^2 + br\sigma_{ac}^2 + cr\sigma_{ab}^2 + bcr\sigma_a^2$$

$$SSAB = cr \sum_i \sum_j (\bar{Y}_{ij\dots} - \bar{Y}_{i\dots} - \bar{Y}_{\cdot j\dots} + \bar{Y}_{\dots\dots})^2 \qquad E[MSAB] = \sigma^2 + r\sigma_{abc}^2 + cr\sigma_{ab}^2$$

$$SSABC = r \sum_i \sum_j \sum_k (\bar{Y}_{ijk\cdot} - \bar{Y}_{ij\dots} - \bar{Y}_{i\cdot k\cdot} - \bar{Y}_{\cdot jk\cdot} + \bar{Y}_{i\dots} + \bar{Y}_{\cdot j\dots} + \bar{Y}_{\dots k\cdot} - \bar{Y}_{\dots\dots})^2$$

$$E[MSABC] = \sigma^2 + r\sigma_{abc}^2$$

$$SSE = \sum_i \sum_j \sum_k \sum_l (Y_{ijkl} - \bar{Y}_{ijk\cdot})^2 \qquad E[MSE] = \sigma^2$$

ANOVA

SOURCE	df	E[MS]
A	$a-1$	$\sigma^2 + r\sigma_{abc}^2 + br\sigma_{ac}^2 + cr\sigma_{ab}^2 + bcr\sigma_a^2$
B	$b-1$	$\sigma^2 + r\sigma_{abc}^2 + ar\sigma_{bc}^2 + cr\sigma_{ab}^2 + acr\sigma_b^2$
C	$c-1$	$\sigma^2 + r\sigma_{abc}^2 + ar\sigma_{bc}^2 + br\sigma_{ac}^2 + abr\sigma_c^2$
AB	$(a-1)(b-1)$	$\sigma^2 + r\sigma_{abc}^2 + cr\sigma_{ab}^2$
AC	$(a-1)(c-1)$	$\sigma^2 + r\sigma_{abc}^2 + br\sigma_{ac}^2$
BC	$(b-1)(c-1)$	$\sigma^2 + r\sigma_{abc}^2 + ar\sigma_{bc}^2$
ABC	$(a-1)(b-1)(c-1)$	$\sigma^2 + r\sigma_{abc}^2$
ERROR	$abc(r-1)$	σ^2
TOTAL	$abcr-1$	—

TESTING FOR 3-FACTOR INTERACTION

$H_0: \sigma^2_{abc} = 0 \qquad H_A: \sigma^2_{abc} > 0$

T.S. $F_{ABC} = \dfrac{MSABC}{MSE}$

RR: $F_{ABC} \geq F_{\alpha, (a-1)(b-1)(c-1), abc(r-1)}$

P-VALUE: $P(F \geq F_{ABC} \mid F \sim F_{(a-1)(b-1)(c-1), abc(r-1)})$

Testing for 2-FACTOR INTERACTIONS

$H_0: \sigma^2_{ab} = 0 \qquad H_A: \sigma^2_{ab} > 0$

T.S. $F_{AB} = \dfrac{MSAB}{MSABC}$

RR: $F_{AB} \geq F_{\alpha, (a-1)(b-1), (a-1)(b-1)(c-1)}$

P-VALUE: $P(F \geq F_{AB} \mid F \sim F_{(a-1)(b-1), (a-1)(b-1)(c-1)})$

(obvious changes for testing σ^2_{bc}, σ^2_{ac})

Testing for ~~3-FAC~~ MAIN EFFECTS

NO CLEAR "DENOMINATOR" FOR MSA, MSB, or MSC

$$\begin{array}{lllllll}
E[MSA]: & \sigma^2 & + r\sigma^2_{abc} & + br\sigma^2_{ac} & + cr\sigma^2_{ab} & + bcr\sigma^2_a \\
E[MSAB]: & \sigma^2 & + r\sigma^2_{abc} & & + cr\sigma^2_{ab} & \\
E[MSAC]: & \sigma^2 & + r\sigma^2_{abc} & + br\sigma^2_{ac} & & \\
E[MSABC]: & \sigma^2 & + r\sigma^2_{abc} & & & \\
\hline
E[MSA - MSAB - MSAC + MSABC]: & 0 & + 0 & + 0 & + 0 & + bcr\sigma^2_a
\end{array}$$

$$\frac{E[MSA + MSABC]}{E[MSAB + MSAC]} = \frac{2\sigma^2 + 2r\sigma^2_{abc} + br\sigma^2_{ac} + cr\sigma^2_{ab} + bcr\sigma^2_a}{2\sigma^2 + 2r\sigma^2_{abc} + br\sigma^2_{ac} + cr\sigma^2_{ab}}$$

SATTERTHWAITE'S APPROXIMATION FOR DEGREES OF FREEDOM FOR "SYNTHETIC F-TEST" EQ 5.27 on P. 169

$M_1 = MSA + MSABC$ $\qquad$ $M_2 = MSAB + MSAC$

$$\nu_1^A = \frac{M_1^2}{\frac{MSA^2}{a-1} + \frac{MSABC^2}{(a-1)(b-1)(c-1)}} \qquad \nu_2^A = \frac{M_2^2}{\frac{MSAB^2}{(a-1)(b-1)} + \frac{MSAC^2}{(a-1)(c-1)}}$$

$H_0: \sigma_a^2 = 0 \qquad H_A: \sigma_a^2 > 0$

T.S.: $F_A^* = \dfrac{MSA + MSABC}{MSAB + MSAC}$

RR: $F_A \geq F_{\alpha, \nu_1^A, \nu_2^A}$

P-VALUE: $P(F \geq F_A \mid F \sim F_{\nu_1^A, \nu_2^A})$

OBVIOUS ADJUSTMENTS FOR σ_b^2, σ_c^2

2-FACTOR MIXED EFFECTS MODEL (A FIXED, B RANDOM)

$$Y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,r \end{array}$$

$$\sum_i \alpha_i = 0$$

$$\left.\begin{array}{l} b_j \sim NID(0,\sigma_b^2) \\ (ab)_{ij} \sim NID(0,\sigma_{ab}^2) \\ e_{ijk} \sim NID(0,\sigma^2) \end{array}\right\} \text{mutually pairwise independent}$$

$$E[Y_{ijk}] = \mu + \alpha_i \qquad V[Y_{ijk}] = \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$$

$$Cov(Y_{ijk}, Y_{ijk'}) = Cov[\mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk},\ \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk'}]$$

$$= Cov(b_j + (ab)_{ij} + e_{ijk},\ b_j + (ab)_{ij} + e_{ijk'})$$

$$= Cov(b_j, b_j) + Cov((ab)_{ij}, (ab)_{ij}) + 0 = \sigma_b^2 + \sigma_{ab}^2$$

$$Cov(Y_{ijk}, Y_{ij'k}) = Cov(b_j + (ab)_{ij} + e_{ijk},\ b_{j'} + (ab)_{ij'} + e_{ij'k}) = 0$$

$$Cov(Y_{ijk}, Y_{i'jk}) = Cov(b_j + (ab)_{ij} + e_{ijk},\ b_j + (ab)_{i'j} + e_{i'jk}) = \sigma_b^2$$

$$\Rightarrow Cov(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} \sigma_b^2 + \sigma_{ab}^2 + \sigma^2 & i=i',\ j=j',\ k=k' \\ \sigma_b^2 + \sigma_{ab}^2 & i=i',\ j=j',\ k\neq k' \\ \sigma_b^2 & i\neq i'\ \ j=j' \\ 0 & j\neq j' \end{cases}$$

$$E[y_{ijk}] = \mu + \alpha_i \qquad V[y_{ijk}] = \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$$

$$\Rightarrow E[y_{ijk}^2] = (\mu+\alpha_i)^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$$

$$E[\bar{y}_{ij.}] = \mu + \alpha_i$$

$$V[\bar{y}_{ij.}] = V\left[\frac{1}{r}\sum_k y_{ijk}\right] = \frac{1}{r^2} V\left[\sum_k y_{ijk}\right]$$

$$= \frac{1}{r^2}\left[\sum_k V(y_{ijk}) + 2\sum\sum_{k<k'} \operatorname{Cov}(y_{ijk}, y_{ijk'})\right]$$

$$= \frac{1}{r^2}\left[r(\sigma_b^2+\sigma_{ab}^2+\sigma^2) + r(r-1)(\sigma_b^2+\sigma_{ab}^2)\right]$$

$$\Rightarrow \boxed{\begin{aligned} E[\bar{y}_{ij.}^2] &= (\mu+\alpha_i)^2 + \frac{1}{r}\left[(\sigma_b^2+\sigma_{ab}^2+\sigma^2)+(r-1)(\sigma_b^2+\sigma_{ab}^2)\right] \\ &= (\mu+\alpha_i)^2 + \frac{1}{r}\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right] \end{aligned}}$$

$$E[\bar{y}_{i..}] = \mu + \alpha_i$$

$$V[\bar{y}_{i..}] = V\left[\frac{1}{br}\sum_j\sum_k y_{ijk}\right] = \left(\frac{1}{br}\right)^2 V\left[\sum_j\sum_k y_{ijk}\right]$$

$$= \left(\frac{1}{br}\right)^2\left[\underbrace{\sum_j\sum_k V(y_{ijk})}_{br \text{ terms}} + \underbrace{2\sum_j\sum\sum_{k<k'} \operatorname{Cov}(y_{ijk}, y_{ijk'})}_{br(r-1)\text{ terms}}\right.$$

$$\left. + \underbrace{\sum\sum_{j\neq j'}\sum_k\sum_{k'} \operatorname{Cov}(y_{ijk}, y_{ij'k'})}_{b(b-1)r^2 \text{ terms}}\right]$$

$br + br(r-1) = br^2$

$br^2 + b(b-1)r^2 = b^2r^2$ ✓

$$= \left(\frac{1}{br}\right)^2\left[br(\sigma_b^2+\sigma_{ab}^2+\sigma^2) + br(r-1)(\sigma_b^2+\sigma_{ab}^2)\right]$$

$$= \left(\frac{1}{br}\right)^2\left[br^2\sigma_b^2 + br^2\sigma_{ab}^2 + br\sigma^2\right] = \frac{1}{br}\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]$$

$$\Rightarrow \boxed{E\left[\bar{Y}_{i..}^2\right] = (\mu+\alpha_i)^2 + \frac{1}{br}\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]}$$

$E[\bar{Y}_{.j.}] = \mu$ (summed over i; $\sum \alpha_i = 0$)

$$V[\bar{Y}_{.j.}] = V\left[\frac{1}{ar}\sum_i\sum_k Y_{ijk}\right] = \left(\frac{1}{ar}\right)^2 V\left[\sum_i\sum_k Y_{ijk}\right]$$

$$= \left(\frac{1}{ar}\right)^2\Bigg[\underbrace{\sum_i\sum_k V(Y_{ijk})}_{ar \text{ terms}} + \underbrace{\sum\sum_{i\neq i'}\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'jk'})}_{a(a-1)r^2 \text{ terms}}$$

$$+ 2\underbrace{\sum_i\sum\sum_{k<k'} Cov(Y_{ijk}, Y_{ijk'})}_{ar(r-1) \text{ terms}}\Bigg]$$

ar
$+ar(r-1)$
ar^2
$+a(a-1)r^2$
$= a^2r^2$ ✓

$$= \left(\frac{1}{ar}\right)^2\Big[ar(\sigma_b^2+\sigma_{ab}^2+\sigma^2) + a(a-1)r^2\sigma_b^2$$

$$+ ar(r-1)(\sigma_b^2+\sigma_{ab}^2)$$

$$\cancel{= \frac{1}{ar}\left[\left[\;+a(a-1)r^2+ar(r-1)\right]\sigma_b^2 + ar(r-1)\sigma_{ab}^2\right.}$$

$$= \frac{1}{ar}\left[\sigma_b^2\left(1+(a-1)r+(r-1)\right) + \sigma_{ab}^2\left(1+(r-1)\right) + \sigma^2\right]$$

$$= \frac{1}{ar}\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]$$

$$\Rightarrow \boxed{E\left[\bar{Y}_{.j.}^2\right] = \mu^2 + \frac{1}{ar}\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]}$$

$E[\bar{Y}_{...}] = \mu$

$$V[\bar{Y}_{...}] = V\left[\frac{1}{abr}\sum_i\sum_j\sum_k Y_{ijk}\right] = \left(\frac{1}{abr}\right)^2 V\left[\sum_i\sum_j\sum_k Y_{ijk}\right]$$

$$= \left(\frac{1}{abr}\right)^2 \Bigg[\underbrace{\sum_i\sum_j\sum_k V(Y_{ijk})}_{abr\ \text{terms}} + \underbrace{2\sum_i\sum_j\sum_{k<k'}\sum Cov(Y_{ijk}, Y_{ijk'})}_{abr(r-1)\ \text{terms}}$$

$$+ \underbrace{\sum_{i\neq i'}\sum\sum_j\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'jk'})}_{a(a-1)br^2\ \text{terms}} + \underbrace{\sum_i\sum_{i'}\sum_{j\neq j'}\sum\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'j'k'})}_{a^2b(b-1)r^2\ \text{terms}}\Bigg]$$

$abr +$
$abr(r-1)$
abr^2
$+ a(a-1)br^2$
a^2br^2
$+ a^2b(b-1)r^2$
$a^2b^2r^2$

$$= \left(\frac{1}{abr}\right)^2 \Big[abr(\sigma_b^2 + \sigma_{ab}^2 + \sigma^2) + abr(r-1)(\sigma_b^2 + \sigma_{ab}^2)$$

$$+ a(a-1)br^2\sigma_b^2 + a^2b(b-1)r^2(0)\Big]$$

$$= \left(\frac{1}{abr}\right)\Big[\sigma_b^2(1 + (r-1) + (a-1)r) + \sigma_{ab}^2(1 + (r-1)) + \sigma^2(1)\Big]$$

$$= \frac{1}{abr}\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]$$

$$\Rightarrow \boxed{E[\bar{Y}_{...}^2] = \mu^2 + \frac{1}{abr}\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]}$$

$$SSE = \sum_i \sum_j \sum_k y_{ijk}^2 - r \sum_i \sum_j \bar{y}_{ij.}^2$$

$$\Rightarrow E[SSE] = \left\{ br\left[\sum_i (\mu+\alpha_i)^2\right] + abr\left(\sigma_b^2 + \sigma_{ab}^2 + \sigma^2\right)\right\}$$

$$- r\left\{ b\sum_i (\mu+\alpha_i)^2 + ab\left(\frac{1}{r}\right)\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]\right\}$$

$$= \sum_i (\mu+\alpha_i)^2 \left\{br - br\right\} + \sigma_b^2\left[abr - abr\right] + \sigma_{ab}^2\left[abr - abr\right]$$

$$+ \sigma^2\left[abr - ab\right] = ab(r-1)\sigma^2$$

$$\Rightarrow \boxed{E(MSE) = E\left(\frac{SSE}{ab(r-1)}\right) = \sigma^2}$$

$$SSAB = \underset{①}{r\sum_i\sum_j \bar{y}_{ij.}^2} - \underset{②}{rb\sum_i \bar{y}_{i..}^2} - \underset{③}{ab\sum_j \bar{y}_{.j.}^2} + \underset{④}{abr\,\bar{y}_{...}^2}$$

Taking Expectations:

$$① = r\left\{b\sum_i (\mu+\alpha_i)^2 + ab\left(\frac{1}{r}\right)\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]\right\}$$

$$② = rb\left\{\sum_i (\mu+\alpha_i)^2 + a\left(\frac{1}{br}\right)\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]\right\}$$

$$③ = a\left\{b\mu^2 + b\left(\frac{1}{ar}\right)\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]\right\}$$

$$④ = abr\left\{\mu^2 + \frac{1}{abr}\left[ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right]\right\}$$

$$① - ② - ③ + ④ =$$

$$\sigma_b^2\left[\overbrace{abr - ar - abr + ar}^{0}\right] + \sigma_{ab}^2\left[abr - ar - br + r\right]$$

$$+ \sigma^2\left[ab - a - b + 1\right] = r(a-1)(b-1)\sigma_{ab}^2 + (a-1)(b-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSAB] = E\left(\frac{SSAB}{(a-1)(b-1)}\right) = r\sigma_{ab}^2 + \sigma^2}$$

$$SSA = br \sum_i \bar{y}_{i..}^2 - abr\,\bar{y}_{...}^2$$

$$E[SSA] = \left\{ br \sum_i (\mu+\alpha_i)^2 + abr \left[\frac{1}{br} \left(r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2 \right) \right] \right\}$$

$$- abr \left\{ \mu^2 + \frac{1}{abr} \left[ar\,\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2 \right] \right\}$$

$$= br \left\{ a\mu^2 + 2\cancel{\sum_i \alpha_i} + \sum_i \alpha_i^2 \right\} - abr\,\mu^2$$

$$+ \sigma_b^2 \left(\cancel{ar - ar} \right] + \sigma_{ab}^2 \left[ar - r \right] + \sigma^2 \left[a - 1 \right]$$

$$= br \sum_i \alpha_i^2 + \sigma_{ab}^2 \left(r(a-1) \right) + \sigma^2 (a-1)$$

$$\Rightarrow \boxed{E[MSA] = E\left(\frac{SSA}{a-1}\right] = \sigma^2 + r\sigma_{ab}^2 + \frac{br \sum_i \alpha_i^2}{a-1}}$$

$$SSB = ar \sum_j \bar{y}_{.j.}^2 - abr\,\bar{y}_{...}^2$$

$$\Rightarrow E[SSB] = \left[abr\mu^2 + abr \left(\frac{1}{ar} \right) \left(ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2 \right) \right]$$

$$- abr \left\{ \mu^2 + \frac{1}{abr} \left[ar\,\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2 \right] \right\}$$

$$= \mu^2 (abr - abr) + \sigma_b^2 (abr - ar) + \sigma_{ab}^2 [br - r] + \sigma^2 [b-1]$$

$$= ar(b-1)\sigma_b^2 + r(b-1)\sigma_{ab}^2 + (b-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSB] = E\left[\frac{SSB}{b-1}\right] = ar\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2}$$

ANOVA – 2 Factor Mixed Effects Model

Source	df	E[MS]
A (Fixed)	$a-1$	$\sigma^2 + r\sigma_{ab}^2 + \frac{br\sum \alpha_i^2}{a-1}$
B (RANDOM)	$b-1$	$\sigma^2 + r\sigma_{ab}^2 + ar\sigma_b^2$
A*B (RANDOM)	$(a-1)(b-1)$	$\sigma^2 + r\sigma_{ab}^2$
ERROR	$ab(r-1)$	σ^2
TOTAL	$abr-1$	—

Test for Interaction

$H_0: \sigma_{ab}^2 = 0 \qquad H_A: \sigma_{ab}^2 > 0$

T.S. $F_{AB} = \frac{MSAB}{MSE}$

RR: $F_{AB} \geq F_{\alpha,(a-1)(b-1),\, ab(r-1)}$

P-VALUE: $P(F \geq F_{AB}) \; F \sim F_{(a-1)(b-1),\, ab(r-1)}$

Tests for Main Effects

FACTOR A (FIXED)

$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$

H_A: NOT ALL $\alpha_i = 0$

T.S. $F_A = \frac{MSA}{MSAB}$

RR: $F_A \geq F_{\alpha, a-1, (a-1)(b-1)}$

Factor B (Random)

$H_0: \sigma_b^2 = 0 \qquad H_A: \sigma_b^2 > 0$

T.S. $F_B = \frac{MSB}{MSAB}$

RR: $F_B \geq F_{\alpha, b-1, (a-1)(b-1)}$

STD ERRORS FOR FIXED FACTOR MEANS

FACTOR A @ Levels i & i' in Factor B @ j

See P. 7.15

$$V[\bar{y}_{ij.}] = \frac{1}{r^2}\left[r(\sigma_b^2 + \sigma_{ab}^2 + \sigma^2) + r(r-1)(\sigma_b^2 + \sigma_{ab}^2)\right]$$

$$= \frac{1}{r^2}\left[\sigma_b^2(r + r^2 - r) + \sigma_{ab}^2(r + r^2 - r) + \sigma^2(r)\right]$$

$$= \sigma_b^2 + \sigma_{ab}^2 + \frac{\sigma^2}{r} = V[\bar{y}_{i'j.}]$$

$$Cov[\bar{y}_{ij.}, \bar{y}_{i'j.}] = Cov\left[\frac{1}{r}\sum_k \bar{y}_{ijk}, \frac{1}{r}\sum_k \bar{y}_{i'jk}\right]$$

$$= \frac{1}{r^2}\sum_k\sum_{k'} Cov(y_{ijk}, y_{i'jk'}) = \frac{1}{r^2}(r^2\sigma_b^2) = \sigma_b^2$$

$$\Rightarrow Cov[\bar{y}_{ij.} - \bar{y}_{i'j.}] = 2\left[\sigma_b^2 + \sigma_{ab}^2 + \frac{\sigma^2}{r}\right] - 2\sigma_b^2$$

$$= 2\left[\sigma_{ab}^2 + \frac{\sigma^2}{r}\right] = \boxed{2\left[\frac{\sigma^2 + r\sigma_{ab}^2}{r}\right] = \sigma^2_{\{\bar{y}_{ij.} - \bar{y}_{i'j.}\}}}$$

NOTE: $E[MSAB] = \sigma^2 + r\sigma_{ab}^2 \Rightarrow \boxed{s^2_{\{\bar{y}_{ij.} - \bar{y}_{i'j.}\}} = \frac{2\,MSAB}{r}}$

$(1-\alpha)100\%$ CI for $\mu_i - \mu_{i'}$ w/in jth Level of B (SIMPLE EFFECT)

$$(\bar{y}_{ij.} - \bar{y}_{i'j.}) \pm t_{\frac{\alpha}{2},(a-1)(b-1)}\sqrt{\frac{2\,MSAB}{r}}$$

7.22

MARGINAL EFFECTS OF LEVELS OF FACTOR A

See P. 7.15

$$V[\bar{Y}_{i..}] = \frac{1}{br}\left[r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right] = V[\bar{Y}_{i'..}]$$

$$Cov[\bar{Y}_{i..}, \bar{Y}_{i'..}] = Cov\left[\frac{1}{br}\sum_j\sum_k Y_{ijk}, \frac{1}{br}\sum_j\sum_k Y_{i'jk}\right]$$

$$= \left(\frac{1}{br}\right)^2\left\{\sum_j\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'jk'}) + \sum\sum_{j\neq j'}\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'j'k'})\right\}$$

$$= \left(\frac{1}{br}\right)^2\left\{br^2\sigma_b^2 + b(b-1)r^2(0)\right\}$$

$$= \frac{1}{b}\sigma_b^2$$

$$\Rightarrow V[\bar{Y}_{i..} - \bar{Y}_{i'..}] = V(\bar{Y}_{i..}) + V(\bar{Y}_{i'..}) - 2Cov(\bar{Y}_{i..}, \bar{Y}_{i'..})$$

$$= 2\left\{\frac{1}{br}\left(r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right] - \frac{1}{b}\sigma_b^2\right\}$$

$$= 2\left\{\sigma_b^2\left(\frac{1}{b} - \frac{1}{b}\right) + \frac{1}{b}\sigma_{ab}^2 + \frac{1}{br}\sigma^2\right\} = 2\left\{\frac{r\sigma_{ab}^2 + \sigma^2}{br}\right\}$$

$$\Rightarrow \boxed{\sigma^2_{\{\bar{Y}_{i..} - \bar{Y}_{i'..}\}} = 2\left\{\frac{r\sigma_{ab}^2 + \sigma^2}{br}\right\}}$$

$$\Rightarrow \boxed{s^2_{\{\bar{Y}_{i..} - \bar{Y}_{i'..}\}} = 2\left\{\frac{MSAB}{br}\right\}}$$

$$\left(E(MSAB) = r\sigma_{ab}^2 + \sigma^2\right)$$

$(1-\alpha)100\%$ CI for $\mu_i - \mu_{i'}$ (MAIN EFFECT)

$$\left(\bar{Y}_{i..} - \bar{Y}_{i'..}\right) \pm t_{\frac{\alpha}{2}, (a-1)(b-1)}\sqrt{\frac{2\,MSAB}{br}}$$

3-FACTOR MIXED EFFECTS MODEL

SOURCE	A FIXED, B/C RANDOM (*)	A/B FIXED, C RANDOM (*)
A	$\sigma^2 + r\sigma^2_{abc} + rc\sigma^2_{ab} + rb\sigma^2_{ac} + rbc\theta^2_a$	$\sigma^2 + r\sigma^2_{abc} + rb\sigma^2_{ac} + rbc\theta^2_a$
B	$\sigma^2 + r\sigma^2_{abc} + rc\sigma^2_{ab} + ra\sigma^2_{bc} + rac\sigma^2_b$	$\sigma^2 + r\sigma^2_{abc} + ra\sigma^2_{bc} + rac\theta^2_b$
C	$\sigma^2 + r\sigma^2_{abc} + rb\sigma^2_{ac} + ra\sigma^2_{bc} + rab\sigma^2_c$	$\sigma^2 + r\sigma^2_{abc} + rb\sigma^2_{ac} + ra\sigma^2_{bc} + rab\sigma^2_c$
AB	$\sigma^2 + r\sigma^2_{abc} + rc\sigma^2_{ab}$	$\sigma^2 + r\sigma^2_{abc} + rc\theta^2_{ab}$
AC	$\sigma^2 + r\sigma^2_{abc} + rb\sigma^2_{ac}$	$\sigma^2 + r\sigma^2_{abc} + rb\sigma^2_{ac}$
BC	$\sigma^2 + r\sigma^2_{abc} + ra\sigma^2_{bc}$	$\sigma^2 + r\sigma^2_{abc} + ra\sigma^2_{bc}$
ABC	$\sigma^2 + r\sigma^2_{abc}$	$\sigma^2 + r\sigma^2_{abc}$
ERROR	σ^2	σ^2

TEST	Test STATISTIC	TEST STATISTIC
ABC	$F_{ABC} = \dfrac{MSABC}{MSE}$	$F_{ABC} = \dfrac{MSABC}{MSE}$
AB	$F_{AB} = \dfrac{MSAB}{MSABC}$	$F_{AB} = \dfrac{MSAB}{MSABC}$
AC	$F_{AC} = \dfrac{MSAC}{MSABC}$	$F_{AC} = \dfrac{MSAC}{MSABC}$
BC	$F_{BC} = \dfrac{MSBC}{MSABC}$	$F_{BC} = \dfrac{MSBC}{MSABC}$
A	$F_A = \dfrac{MSA + MSABC}{MSAB + MSAC}$ (**)	$F_A = \dfrac{MSA}{MSAC}$
B	$F_B = \dfrac{MSB + MSABC}{MSAB + MSAC}$ (**)	$F_B = \dfrac{MSB}{MSBC}$
C	$F_C = \dfrac{MSC + MSABC}{MSAC + MSBC}$ (**)	$F_C = \dfrac{MSC + MSABC}{MSAC + MSBC}$ (**)

(*) $\theta^2_a = \dfrac{\Sigma\alpha_i^2}{a-1}$ $\quad \theta^2_b = \dfrac{\Sigma\beta_j^2}{b-1}$ $\quad \theta^2_{ab} = \dfrac{\Sigma\Sigma(\alpha\beta)^2_{ij}}{(a-1)(b-1)}$

(**) use Satterthwaite's approx. for df. (Eq 5.27)

NESTED FACTOR DESIGNS

Levels of FACTOR B ARE "NESTED" UNDER LEVELS OF FACTOR A.

EXAMPLE -

① A & B "FIXED"

A - SODA BRAND (COKE vs PEPSI)

B - RESTAURANT ESTABLISHMENT (ESTABLISHMENTS "ALIGN" WITH BRANDS)

Y ≡ CAFFEINE CONTENT

② A FIXED - B RANDOM

A - REGION (NE, MW, S, W)

B - SCHOOL DISTRICT (SCHOOL DISTRICTS WITHIN REGIONS)

Y ≡ Teacher's salaries

③ A RANDOM - B RANDOM...

A - YARN PRODUCER (MANY SMALL PRODUCERS, FEW SAMPLED)

B - BATCH OF YARN (SEVERAL ROLLS SAMPLED FROM EACH PRODUCER)

STATISTICAL MODELS

① A & B FIXED

$$Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + e_{k(ij)} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \end{array} \qquad e_{k(ij)} \sim NID(0,\sigma^2)$$

$$\sum_j \beta_{j(i)} = 0 \;\; \forall i \qquad \sum_i \alpha_i = 0$$

(2) A FIXED / B RANDOM

$$Y_{ijk} = \mu + \alpha_i + b_{j(i)} + e_{k(ij)} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \end{array}$$

$e_{k(ij)} \sim NID(0,\sigma^2)$ $\quad b_{j(i)} \sim NID(0,\sigma^2_{b(a)})$ $\quad \{e\} \perp \{b\}$

$\sum_i \alpha_i = 0$

(3) A & B RANDOM

$$Y_{ijk} = \mu + a_i + b_{j(i)} + e_{k(ij)} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \end{array}$$

$a_i \sim NID(0,\sigma^2_a)$ $\quad b_{j(i)} \sim NID(0,\sigma^2_{b(a)})$ $\quad e_{k(i,j)} \sim NID(0,\sigma^2)$

$\{a_i\} \perp \{b_{j(i)}\} \perp \{e_{k(ij)}\}$ $\quad$ ($\sigma^2 = \sigma^2_{c(B)}$ in Kuehl's notation)

SUMS OF SQUARES

$$SSA = \sum_i\sum_j\sum_k (\bar{Y}_{i..} - \bar{Y}_{...})^2 = bc\sum_i(\bar{Y}_{i..}-\bar{Y}_{...})^2 \qquad df_A = a-1$$

$$SSB(A) = \sum_i\sum_j\sum_k (\bar{Y}_{ij.} - \bar{Y}_{i..})^2 = c\sum_i\sum_j(\bar{Y}_{ij.}-\bar{Y}_{i..})^2 \qquad df_{B(A)} = a(b-1)$$

$$SSE = \sum_i\sum_j\sum_k (Y_{ijk} - \bar{Y}_{ij.})^2 = (c-1)\sum_i\sum_j s^2_{ij} \qquad df_E = ab(c-1)$$

NOTE: $SSA = bc\sum_i \bar{Y}^2_{i..} - abc\,\bar{Y}^2_{...}$

$SSB(A) = c\sum_i\sum_j \bar{Y}^2_{ij.} - bc\sum \bar{Y}^2_{i..}$

$SSE = \sum_i\sum_j\sum_k Y^2_{ijk} - c\sum_i\sum_j \bar{Y}^2_{ij.}$

(1) A FIXED, B FIXED

$$E[Y_{ijk}] = \mu + \alpha_i + \beta_{j(i)} \qquad V[Y_{ijk}] = \sigma^2$$

$$E[Y_{ijk}^2] = [\mu + \alpha_i + \beta_{j(i)}]^2 + \sigma^2$$

$$E\left\{\sum_i\sum_j\sum_k Y_{ijk}^2\right\} = \left\{\sum_i\sum_j\sum_k\left[\mu^2 + \alpha_i^2 + \beta_{j(i)}^2 + 2\mu\alpha_i + 2\mu\beta_{j(i)} + 2\alpha_i\beta_{j(i)}\right] + abc\sigma^2\right\}$$

$$= abc\mu^2 + bc\sum_i\alpha_i^2 + c\sum_i\sum_j\beta_{j(i)}^2 + 2bc\mu\underbrace{\cancel{\sum_i\alpha_i}}_{0} + 2c\mu\underbrace{\cancel{\sum_i\sum_j\beta_{j(i)}}}_{0} + 2c\underbrace{\cancel{\sum_j\beta_{j(i)}}}_{0}\underbrace{\cancel{\sum_i\alpha_i}}_{0} + abc\sigma^2$$

$$= \boxed{abc\mu^2 + bc\sum_i\alpha_i^2 + c\sum_i\sum_j\beta_{j(i)}^2 + abc\sigma^2 = E\left(\sum_i\sum_j\sum_k Y_{ijk}^2\right)}$$

$$E[\bar{Y}_{ij.}] = \mu + \alpha_i + \beta_{j(i)} \qquad V[\bar{Y}_{ij.}] = \frac{\sigma^2}{c}$$

$$E[\bar{Y}_{ij.}^2] = [\mu + \alpha_i + \beta_{j(i)}]^2 + \frac{\sigma^2}{c}$$

$$E\left[c\sum_i\sum_j\bar{Y}_{ij.}^2\right] = c\sum_i\sum_j(\mu + \alpha_i + \beta_{j(i)})^2 + ab\sigma^2$$

$$= abc\mu^2 + bc\sum_i\alpha_i^2 + c\sum_i\sum_j\beta_{j(i)}^2 + 2bc\underbrace{\cancel{\sum_i\mu\alpha_i}}_{0} + 2c\mu\underbrace{\cancel{\sum_i\sum_j\beta_{j(i)}}}_{0} + 2c\underbrace{\cancel{\sum_j\beta_{j(i)}}}_{0}\underbrace{\cancel{\sum_i\alpha_i}}_{0} + ab\sigma^2$$

$$= \boxed{abc\mu^2 + bc\sum_i\alpha_i^2 + c\sum_i\sum_j\beta_{j(i)}^2 + ab\sigma^2 = E\left[c\sum_i\sum_j\bar{Y}_{ij.}^2\right]}$$

A & B FIXED

$$E[\bar{y}_{i..}] = \mu + \alpha_i + \sum_j \cancel{\beta_{j(i)}}^{0} = \mu + \alpha_i$$

$$V[\bar{y}_{i..}] = \frac{\sigma^2}{bc}$$

$$E[\bar{y}_{i..}^2] = (\mu + \alpha_i)^2 + \frac{\sigma^2}{bc}$$

$$E\left[bc \sum_i \bar{y}_{i..}^2\right] = bc \sum_i (\mu + \alpha_i)^2 + abc\frac{\sigma^2}{bc}$$

$$= bc \sum_i (\mu^2 + 2\alpha_i + \alpha_i^2) + c\sigma^2$$

$$= abc\mu^2 + 2bc \cancel{\sum \alpha_i}^{0} + bc \sum_i \alpha_i^2 + a\sigma^2$$

$$= \boxed{abc\mu^2 + bc \sum_i \alpha_i^2 + a\sigma^2 = E\left[bc \sum_i \bar{y}_{i..}^2\right]}$$

$$E[\bar{y}_{...}] = \mu + \sum_i \cancel{\alpha_i}^{0} + \sum_i \sum_j \cancel{\beta_{j(i)}}^{0} = \mu$$

$$V[\bar{y}_{...}] = \frac{\sigma^2}{abc}$$

$$E[\bar{y}_{...}^2] = \mu^2 + \frac{\sigma^2}{abc} \Rightarrow \boxed{E[abc\bar{y}_{...}^2] = abc\mu^2 + \sigma^2}$$

$$\Rightarrow E[SSA] = \left[abc\mu^2 + bc \sum_i \alpha_i^2 + a\sigma^2\right] - \left(abc\mu^2 + \sigma^2\right)$$

$$= bc \sum \alpha_i^2 + (a-1)\sigma^2 \Rightarrow \boxed{E[MSA] = \frac{bc \sum_i \alpha_i^2}{a-1} + \sigma^2}$$

$$E(SSB(A)] = \left[abc\mu^2 + bc \sum_i \alpha_i^2 + c \sum_i \sum_j \beta_{j(i)}^2 + ab\sigma^2\right] - \left[abc\mu^2 + bc \sum_i \alpha_i^2 + a\sigma^2\right]$$

$$= c \sum_i \sum_j \beta_{j(i)}^2 + \sigma^2 a(b-1) \Rightarrow \boxed{E(MSB(A)] = \frac{c \sum_i \sum_j \beta_{j(i)}^2}{a(b-1)} + \sigma^2}$$

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
AMPAD

A : B FIXED

$$E[SSE] = \left[ab\cancel{c}\mu^2 + bc\sum_i \cancel{\alpha_i}^2 + c\sum_i\sum_j \cancel{\beta}_{j(i)}^2 + abc\sigma^2\right]$$

$$- \left[ab\cancel{c}\mu^2 + bc\sum_i \alpha_i^2 + c\sum_i\sum_j \beta_{j(i)}^2 + ab\sigma^2\right]$$

$$= ab(c-1)\sigma^2 \implies \boxed{E[MSE] = \sigma^2}$$

ANOVA (A : B FIXED)

SOURCE	df	SS	E[MS]
A	$a-1$	$bc\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2$	$\frac{bc\sum\alpha_i^2}{a-1}+\sigma^2$
B(A)	$a(b-1)$	$c\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..})^2$	$\frac{c\sum_i\sum_j\beta_{j(i)}^2}{a(b-1)}+\sigma^2$
ERROR (C(B(A))	$ab(c-1)$	$\sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{ij.})^2$	σ^2
TOTAL	$abc-1$	$\sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{...})^2$	

Nested Factor Effect

$H_0: \beta_{j(i)} = 0 \quad \forall\ i, j$ $\qquad$ H_A: NOT ALL $\beta_{j(i)} = 0$

T.S. $F_{B(A)} = \frac{MSB(A)}{MSE}$ $\qquad$ RR: $F_{B(A)} \geq F_{\alpha,\, a(b-1),\, ab(c-1)}$

Nesting Factor Effect

$H_0: \alpha_i = 0 \quad \forall\ i$ $\qquad$ H_A: Not all $\alpha_i = 0$

T.S. $F_A = \frac{MSA}{MSE}$ $\qquad$ RR: $F_A \geq F_{\alpha,\, a-1,\, ab(c-1)}$

Comparing Levels of B within levels of A

$$E[\bar{Y}_{ij} - \bar{Y}_{ij'}] = [\mu + \alpha_i + \beta_{j(i)}] - [\mu + \alpha_i + \beta_{j'(i)}]$$

$$= \beta_{j(i)} - \beta_{j'(i)}$$

$$V[\bar{Y}_{ij.} - \bar{Y}_{ij'.}] = \frac{2\sigma^2}{c} = \sigma^2_{\{\bar{Y}_{ij.} - \bar{Y}_{ij'.}\}}$$

$$\Rightarrow S^2_{\{\bar{Y}_{ij.} - \bar{Y}_{ij'.}\}} = \frac{2MSE}{c}$$

$(1-\alpha)100\%$ CI for $\beta_{j(i)} - \beta_{j'(i)}$

$$(\bar{Y}_{ij.} - \bar{Y}_{ij'.}) \pm t_{\frac{\alpha}{2}, ab(c-1)} \sqrt{\frac{2MSE}{c}}$$

Comparing Levels of Factor A

$$E[\bar{Y}_{i..} - \bar{Y}_{i'..}] = [\mu + \alpha_i + \sum_j \cancel{\beta_{j(i)}}^{0}] - [\mu + \alpha_{i'} + \sum_j \cancel{\beta_{j(i')}}^{0}]$$

$$= \alpha_i - \alpha_{i'}$$

$$V[\bar{Y}_{i..} - \bar{Y}_{i'..}] = \frac{2\sigma^2}{bc} \Rightarrow S^2_{\{\bar{Y}_{i..} - \bar{Y}_{i'..}\}} = \frac{2MSE}{bc}$$

$(1-\alpha)100\%$ CI for $\alpha_i - \alpha_{i'}$

$$(\bar{Y}_{i..} - \bar{Y}_{i'..}) \pm t_{\frac{\alpha}{2}, ab(c-1)} \sqrt{\frac{2MSE}{bc}}$$

② A FIXED / B RANDOM

$$Y_{ijk} = \mu + \alpha_i + b_{j(i)} + e_{k(ij)} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \end{array}$$

$$Cov(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} \sigma_b^2 + \sigma^2 & i=i' \quad j=j' \quad k=k' \\ \sigma_b^2 & i=i' \quad j=j' \quad k\neq k' \\ 0 & \text{o.w.} \end{cases}$$

NOTE: σ_b^2 is shorthand for $\sigma_{b(a)}^2$

$$E[\bar{Y}_{\dots}] = \mu$$

$$V[\bar{Y}_{\dots}] = V\Big[\frac{1}{abc}\sum_i\sum_j\sum_k Y_{ijk}\Big]$$

$$= \left(\frac{1}{abc}\right)^2 V\Big[\sum_i\sum_j\sum_k Y_{ijk}\Big]$$

$$= \left(\frac{1}{abc}\right)^2 \Big\{ \sum_i\sum_j\sum_k V(Y_{ijk}) + \sum_i\sum_j\sum\sum_{k\neq k'} Cov(Y_{ijk}, Y_{ijk'})$$

$$+ \sum\sum_{i\neq i'}\sum\sum_{j\,j'}\sum\sum_{k\,k'} \overbrace{Cov(Y_{ijk}, Y_{i'j'k'})}^{0}$$

$$+ \sum_i \sum\sum_{j\neq j'} \sum\sum_{k\,k'} \overbrace{Cov(Y_{ijk}, Y_{i'j'k'})}^{0} \Big\}$$

$$= \left(\frac{1}{abc}\right)^2 \Big\{ abc(\sigma_b^2 + \sigma^2) + abc(c-1)\sigma_b^2 \Big\}$$

$$= \frac{1}{abc}\Big\{ \sigma_b^2 + \sigma^2 + (c-1)\sigma_b^2 \Big\} = \frac{1}{abc}\Big\{\sigma^2 + c\sigma_b^2\Big\}$$

$$\Rightarrow E[\bar{Y}_{\dots}^2] = \mu^2 + \frac{1}{abc}\Big\{\sigma^2 + c\sigma_b^2\Big\}$$

$$\Rightarrow \boxed{E[abc\,\bar{Y}_{\dots}^2] = abc\,\mu^2 + \Big\{\sigma^2 + c\sigma_b^2\Big\}}$$

A FIXED / B RANDOM

$$E[\bar{Y}_{i..}] = \mu + \alpha_i$$

$$V[\bar{Y}_{i..}] = V\left[\frac{1}{bc} \sum_j \sum_k Y_{ijk}\right] = \left(\frac{1}{bc}\right)^2 V\left[\sum_j \sum_k Y_{ijk}\right]$$

$$= \left(\frac{1}{bc}\right)^2 \Big\{ \sum_j \sum_k V(Y_{ijk}) + \sum_j \sum\sum_{k \neq k'} Cov(Y_{ijk}, Y_{ijk'})$$

$$+ \sum\sum_{j \neq j'} \sum\sum_{k\ k'} \overset{0}{\cancel{Cov(Y_{ijk}, Y_{ij'k'})}} \Big\}$$

$$= \left(\frac{1}{bc}\right)^2 \left\{ bc(\sigma_b^2 + \sigma^2) + bc(c-1)\sigma_b^2 \right\}$$

$$= \frac{1}{bc} \left\{ \sigma^2 + c\sigma_b^2 \right\}$$

$$\Rightarrow E\{\bar{Y}_{i..}^2\} = (\mu + \alpha_i)^2 + \frac{1}{bc}\{\sigma^2 + c\sigma_b^2\}$$

$$\Rightarrow E\{bc \sum_i \bar{Y}_{i..}^2\} = bc\left\{\sum_i (\mu + \alpha_i)^2\right\} + a(\sigma^2 + c\sigma_b^2)$$

$$= bc \sum_i (\mu^2 + \overset{0}{2\cancel{\mu}\alpha_i} + \alpha_i^2) + a(\sigma^2 + c\sigma_b^2)$$

$$= \boxed{abc\mu^2 + bc\sum_i \alpha_i^2 + a(\sigma^2 + c\sigma_b^2) = E\{bc\sum_i \bar{Y}_{i..}^2\}}$$

$$E[\bar{Y}_{ij.}] = \mu + \alpha_i$$

$$V[\bar{Y}_{ij.}] = V\left[\frac{1}{c}\sum_k Y_{ijk}\right] = \frac{1}{c^2} V\left\{\sum_k Y_{ijk}\right\}$$

$$= \frac{1}{c^2}\left\{\sum_k V(Y_{ijk}) + \sum_{k\neq k'}\sum Cov(Y_{ijk}, Y_{ijk'})\right\}$$

$$= \frac{1}{c^2}\left\{c(\sigma^2 + \sigma_b^2) + c(c-1)\sigma_b^2\right\}$$

$$= \frac{1}{c}\left\{\sigma^2 + c\sigma_b^2\right\}$$

$$\Rightarrow E\{\bar{Y}_{ij.}^2\} = (\mu+\alpha_i)^2 + \frac{1}{c}\{\sigma^2 + c\sigma_b^2\}$$

$$\Rightarrow E\left\{c\sum_i\sum_j \bar{Y}_{ij.}^2\right\} = c\sum_i\sum_j\left\{(\mu+\alpha_i)^2 + \frac{1}{c}\{\sigma^2 + c\sigma_b^2\}\right\}$$

$$= c\left\{ab\mu^2 + 2b\cancelto{0}{\sum \alpha_i} + b\sum_i \alpha_i^2\right\} + abc\left(\frac{1}{c}\right)\{\sigma^2 + c\sigma_b^2\}$$

$$= \boxed{abc\mu^2 + bc\sum_i \alpha_i^2 + ab\sigma^2 + abc\sigma_b^2 = E\left\{c\sum_i\sum_j \bar{Y}_{ij.}^2\right\}}$$

$$E[Y_{ijk}] = \mu + \alpha_i$$

$$V[Y_{ijk}] = \sigma_b^2 + \sigma^2 \Rightarrow E\{Y_{ijk}^2\} = (\mu+\alpha_i)^2 + \sigma_b^2 + \sigma^2$$

$$\Rightarrow \sum_i\sum_j\sum_k E[Y_{ijk}^2] =$$

$$= \sum_i\sum_j\sum_k (\mu^2 + \cancelto{0}{2\mu\alpha_i} + \alpha_i^2) + abc(\sigma_b^2 + \sigma^2)$$

$$= \boxed{abc\mu^2 + bc\sum_j \alpha_i^2 + abc\sigma_b^2 + abc\sigma^2 = \sum_i\sum_j\sum_k E(Y_{ijk}^2)}$$

A FIXED/B RANDOM

$$SSA = bc \sum_i \bar{Y}_{i..}^2 - abc\,\bar{Y}_{...}^2 \qquad df_A = a-1$$

$$E\{SSA\} = \{ab\cancel{c}\mu^2 + bc\sum_i \alpha_i^2 + a(\sigma^2 + c\sigma_b^2)\} -$$

$$\{ab\cancel{c}\mu^2 + \sigma^2 + c\sigma_b^2\}$$

$$= bc\sum_i \alpha_i^2 + \sigma^2(a-1) + \sigma_b^2\,c(a-1)$$

$$\Rightarrow \boxed{E[MSA] = \frac{bc\sum_i \alpha_i^2}{a-1} + c\sigma_b^2 + \sigma^2}$$

$$SSB(A) = c\sum_i\sum_j \bar{Y}_{ij.}^2 - bc\sum_i \bar{Y}_{i..}^2 \qquad df_{B(A)} = a(b-1)$$

$$E\{SSB(A)\} = \{ab\cancel{c}\mu^2 + bc\cancel{\sum_i}\alpha_i^2 + ab\sigma^2 + abc\sigma_b^2\}$$

$$- \{ab\cancel{c}\mu^2 + bc\cancel{\sum_i}\alpha_i^2 + a(\sigma^2 + c\sigma_b^2)\}$$

$$= \{\sigma^2(a(b-1)) + \sigma_b^2(ac(b-1))\}$$

$$\Rightarrow \boxed{E[MSB(A)] = \sigma^2 + c\sigma_b^2}$$

$$SSE = \sum_i\sum_j\sum_k Y_{ijk}^2 - c\sum_i\sum_j \bar{Y}_{ij.}^2 \qquad df_E = ab(c-1)$$

$$\Rightarrow E(SSE) = [ab\cancel{c}\mu^2 + bc\cancel{\sum_i}\alpha_i^2 + ab\cancel{c}\sigma_b^2 + abc\sigma^2]$$

$$- [ab\cancel{c}\mu^2 + bc\cancel{\sum_i}\alpha_i^2 + ab\cancel{c}\sigma_b^2 + ab\sigma^2]$$

$$= ab(c-1)\sigma^2 \Rightarrow \boxed{E[MSE] = \sigma^2}$$

ANOVA (A FIXED - B RANDOM)

SOURCE	df	E(MS)
A	$a-1$	$\sigma^2 + c\sigma^2_{b(a)} + \frac{bc\sum_i \alpha_i^2}{a-1}$
B(A)	$a(b-1)$	$\sigma^2 + c\sigma^2_{b(a)}$
ERROR	$ab(c-1)$	σ^2
TOTAL	$abc-1$	

TEST	TEST STATISTIC	RR
A	$F_A = \frac{MSA}{MSB(A)}$	$F_A \geq F_{\alpha, a-1, a(b-1)}$
B(A)	$F_{B(A)} = \frac{MSB(A)}{MSE}$	$F_{B(A)} \geq F_{\alpha, a(b-1), ab(c-1)}$

Test A: $H_0: \alpha_i = 0 \quad \forall\ i = 1, \dots, a$

H_A: Not all $\alpha_i = 0$

TEST B(A): $H_0: \sigma^2_{b(a)} = 0$

$H_A: \sigma^2_{b(a)} > 0$

Comparing Levels of Factor A

$$E[\bar{Y}_{i\cdot\cdot}] = \mu + \alpha_i$$

$$V(\bar{Y}_{i\cdot\cdot}) = \frac{1}{bc}\{\sigma^2 + c\sigma^2_{b(a)}\}$$

$$Cov(\bar{Y}_{i\cdot\cdot}, \bar{Y}_{i'\cdot\cdot}) = Cov\left(\frac{1}{bc}\sum_j\sum_k Y_{ijk}, \frac{1}{bc}\sum_j\sum_k Y_{i'jk}\right) = 0$$

$$\Rightarrow E\{\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}\} = (\mu+\alpha_i) - (\mu+\alpha_{i'}) = \alpha_i - \alpha_{i'}$$

$$V[\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}] = \frac{2}{bc}[\sigma^2 + c\sigma^2_{b(a)}]$$

$$\Rightarrow S^2_{\{\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}\}} = \frac{2\,MSB(A)}{bc}$$

$\Rightarrow$ $(1-\alpha)100\%$ CI for $\alpha_i - \alpha_{i'}$

$$(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}) \pm t_{\frac{\alpha}{2}, a(b-1)}\sqrt{\frac{2\,MSB(A)}{bc}}$$

Bonferroni or Tukey Adjustments for all pairwise comparisons

A RANDOM / B RANDOM

MODEL

$$Y_{ijk} = \mu + a_i + b_{j(i)} + e_{k(ij)} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \end{array}$$

$a_i \sim NID(0,\sigma_a^2) \quad b_{j(i)} \sim NID(0,\sigma^2_{b(a)}) \quad e_{k(ij)} \sim NID(0,\sigma^2)$

$\{a_i\} \perp \{b_{j(i)}\} \perp \{e_{k(ij)}\}$

$E\{Y_{ijk}\} = \mu$

$$Cov\{Y_{ijk}, Y_{i'j'k'}\} = \begin{cases} \sigma_a^2 + \sigma^2_{b(a)} + \sigma^2 & i=i' \; j=j' \; k=k' \\ \sigma_a^2 + \sigma^2_{b(a)} & i=i' \; j=j' \; k\neq k' \\ \sigma_a^2 & i=i' \; j\neq j' \\ 0 & i\neq i' \end{cases}$$

$E\{\bar{Y}_{\dots}\} = \mu$

$$V\{\bar{Y}_{\dots}\} = V\left\{\frac{1}{abc}\sum_i\sum_j\sum_k Y_{ijk}\right\} = \left(\frac{1}{abc}\right)^2 V\left\{\sum_i\sum_j\sum_k Y_{ijk}\right\}$$

$$= \left(\frac{1}{abc}\right)^2 \left\{ \sum_i\sum_j\sum_k V(Y_{ijk}) + \sum_i\sum_j\sum\sum_{k\neq k'} Cov(Y_{ijk}, Y_{ijk'}) \right.$$

$$\left. + \sum_i \sum\sum_{j\neq j'} \sum_k\sum_{k'} Cov(Y_{ijk}, Y_{ij'k'}) + \sum\sum_{i\neq i'} \sum_j\sum_{j'}\sum_k\sum_{k'} Cov(Y_{ijk}, Y_{i'j'k'}) \right\}$$

$$= \left(\frac{1}{abc}\right)^2 \left\{ abc\left(\sigma_a^2 + \sigma^2_{b(a)} + \sigma^2\right) + abc(c-1)\left(\sigma_a^2 + \sigma^2_{b(a)}\right) \right.$$

$$\left. + ab(b-1)c^2\sigma_a^2 + a(a-1)b^2c^2(0) \right\}$$

$$= \left(\frac{1}{abc}\right)^2 \left\{ \sigma_a^2\left(abc + abc(c-1) + ab(b-1)c^2\right) + \sigma^2_{b(a)}\left(abc + abc(c-1)\right) \right.$$

$$\left. + \sigma^2(abc) \right\} =$$

$$= \left(\frac{1}{abc}\right)^2 \left\{ \sigma_a^2 \left(abc^2 + abc^2(b-1)\right) + \sigma_{b(a)}^2\, abc^2 + \sigma^2 abc \right\}$$

$$= \left(\frac{1}{abc}\right)^2 \left\{ \sigma_a^2 \left(ab^2c^2\right) + \sigma_{b(a)}^2\, abc^2 + \sigma^2 (abc) \right\}$$

$$= \frac{1}{abc} \left\{ bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2 \right\} = V(\bar{Y}_{\cdots})$$

$$\Rightarrow E\{\bar{Y}_{\cdots}^2\} = \mu^2 + \frac{1}{abc}\left\{bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2\right\}$$

$$\Rightarrow \boxed{E\{abc\bar{Y}_{\cdots}^2\} = abc\mu^2 + bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2}$$

$$E[\bar{Y}_{i\cdot\cdot}] = \mu$$

$$V[\bar{Y}_{i\cdot\cdot}] = V\left[\frac{1}{bc}\sum_j\sum_k Y_{ijk}\right] = \left(\frac{1}{bc}\right)^2 V\left\{\sum_j\sum_k Y_{ijk}\right\}$$

$$= \left(\frac{1}{bc}\right)^2 \left\{ \sum_j\sum_k V(Y_{ijk}) + \sum_j \sum\sum_{k\neq k'} Cov(Y_{ijk}, Y_{ijk'}) \right.$$

$$\left. + \sum\sum_{j\neq j'} \sum_k\sum_{k'} Cov(Y_{ijk}, Y_{ij'k'}) \right\}$$

$$= \left(\frac{1}{bc}\right)^2 \left\{ bc\left(\sigma_a^2 + \sigma_{b(a)}^2 + \sigma^2\right) + bc(c-1)\left(\sigma_a^2 + \sigma_{b(a)}^2\right) \right.$$

$$\left. + b(b-1)c^2\sigma_a^2 \right\}$$

$$= \left(\frac{1}{bc}\right)\left\{ \sigma_a^2 \overbrace{\left(1 + (c-1) + (b-1)c\right)}^{bc} + \sigma_{b(a)}^2 \overbrace{\left(1 + (c-1)\right)}^{c} + \sigma^2 \right\}$$

$$\Rightarrow E\{\bar{Y}_{i\cdot\cdot}^2\} = \mu^2 + \frac{1}{bc}\left\{bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2\right\}$$

$$\Rightarrow \boxed{E\{bc\sum_i \bar{Y}_{i..}^2\} = abc\mu^2 + abc\sigma_a^2 + ac\sigma_{b(a)}^2 + a\sigma^2}$$

$$E[\bar{Y}_{ij.}] = \mu$$

$$V[\bar{Y}_{ij.}] = V\left[\frac{1}{c}\sum_k Y_{ijk}\right] = \frac{1}{c^2} V\left[\sum_k Y_{ijk}\right]$$

$$= \frac{1}{c^2}\left\{\sum_k V(Y_{ijk}) + \sum\sum_{k \neq k'} Cov(Y_{ijk}, Y_{ijk})\right\}$$

$$= \frac{1}{c^2}\left\{c(\sigma_a^2 + \sigma_{b(a)}^2 + \sigma^2) + c(c-1)(\sigma_a^2 + \sigma_{b(a)}^2)\right\}$$

$$= \frac{1}{c}\left\{(\sigma_a^2 + \sigma_{b(a)}^2 + \sigma^2) + (c-1)(\sigma_a^2 + \sigma_{b(a)}^2)\right)$$

$$= \frac{1}{c}\left\{c\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2\right\} \Rightarrow E\{\bar{Y}_{ij.}^2\} = \mu^2 + \frac{1}{c}(c\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma$$

$$\Rightarrow \boxed{E\{c\sum_i\sum_j \bar{Y}_{ij.}^2\} = abc\mu^2 + abc\sigma_a^2 + abc\sigma_{b(a)}^2 + ab\sigma^2}$$

$$E[Y_{ijk}] = \mu$$

$$V[Y_{ijk}] = \sigma_a^2 + \sigma_{b(a)}^2 + \sigma^2$$

$$\Rightarrow E[Y_{ijk}^2] = \mu^2 + \sigma_a^2 + \sigma_{b(a)}^2 + \sigma^2$$

$$\Rightarrow E\{\sum_i\sum_j\sum_k Y_{ijk}^2\} = abc\mu^2 + abc\sigma_a^2 + abc\sigma_{b(a)}^2 + abc\sigma^2$$

$$SSA = bc \sum_i \bar{Y}_{i..}^2 - abc\,\bar{Y}_{...}^2$$

$$\Rightarrow E[SSA] = \left[ab\cancel{c}\mu^2 + abc\sigma_a^2 + ac\,\sigma_{b(a)}^2 + a\sigma^2\right]$$

$$- \left[a\cancel{b}c\mu^2 + bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2\right]$$

$$= bc(a-1)\sigma_a^2 + c(a-1)\sigma_{b(a)}^2 + (a-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSA] = bc\sigma_a^2 + c\sigma_{b(a)}^2 + \sigma^2}$$

$$SSB(A) = c\sum_i\sum_j \bar{Y}_{ij.}^2 - bc\sum_i \bar{Y}_{i..}^2$$

$$\Rightarrow E[SSB(A)] = \left\{ab\cancel{c}\mu^2 + abc\sigma_a^2 + abc\,\sigma_{b(a)}^2 + ab\sigma^2\right\}$$

$$- \left\{ab\cancel{c}\mu^2 + ab\cancel{c}\,\sigma_a^2 + ac\sigma_{b(a)}^2 + a\sigma^2\right\}$$

$$= ac(b-1)\,\sigma_{b(a)}^2 + a(b-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSB(A)] = c\sigma_{b(a)}^2 + \sigma^2}$$

$$SSE = \sum_i\sum_j\sum_k Y_{ijk}^2 - c\sum_i\sum_j \bar{Y}_{ij.}^2$$

$$\Rightarrow E[SSE] = \left\{ab\cancel{c}\mu^2 + abc\sigma_a^2 + abc\sigma_{b(a)}^2 + abc\,\sigma^2\right\}$$

$$- \left\{ab\cancel{c}\mu^2 + ab\cancel{c}\sigma_a^2 + ab\cancel{c}\sigma_{b(a)}^2 + ab\sigma^2\right\}$$

$$= ab(c-1)\sigma^2 \quad \Rightarrow \boxed{E[MSE] = \sigma^2}$$

ANOVA (A RANDOM / B RANDOM)

SOURCE	df	E[MS]
A	$a-1$	$\sigma^2 + c\sigma^2_{b(a)} + bc\,\sigma^2_a$
B(A)	$a(b-1)$	$\sigma^2 + c\,\sigma^2_{b(a)}$
ERROR	$ab(c-1)$	σ^2
TOTAL	$abc-1$	

Test	Test Statistic	RR
A	$F_A = \dfrac{MSA}{MSB(A)}$	$F_A \geq F_{\alpha, a-1, a(b-1)}$
B(A)	$F_{B(A)} = \dfrac{MSB(A)}{MSE}$	$F_{B(A)} \geq F_{\alpha, a(b-1), ab(c-1)}$

Test A: $H_0: \sigma_a^2 = 0$ $\quad H_A: \sigma_a^2 > 0$

Test B: $H_0: \sigma^2_{b(a)} = 0$ $\quad H_A: \sigma^2_{b(a)} > 0$

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EXAMPLE – CAFFEINE CONTENT (COKE VS PEPSI)

COKE VS PEPSI IN 5 ESTABLISHMENTS

(ESTABLISHMENTS NESTED UNDER BRANDS).

10 REPS/ ESTABLISHMENT

TREATING ESTABLISHMENTS AS RANDOM (Individual LOCATIONS)

$$Y_{ijk} = \mu + \alpha_i + b_{j(i)} + \varepsilon_{k(ij)}$$

$i = 1, 2$ (BRAND)
$j = 1, \dots, 5$ (ESTABLISHMENT)
$k = 1, \dots, 10$ (REPLICATE)

ESTABLISHMENT (BRAND)

BRAND	j=1	j=2	j=3	j=4	j=5	
(i=1) COKE	44.6 (3.4)	39.1 (3.9)	38.2 (3.0)	35.0 (4.1)	32.8 (2.7)	$\overline{Y}_{1..} = 37.94$
(i=2) PEPSI	39.3 (1.5)	38.0 (3.0)	36.7 (1.7)	35.5 (2.7)	35.1 (1.6)	$\overline{Y}_{2..} = 36.92$
← MEANINGLESS	MARGINS →					$\overline{Y}_{...} = 37.43$

$$SSA = bc \sum_{i=1}^{2} (\overline{Y}_{i..} - \overline{Y}_{...})^2 = 5(10)\left[(37.94 - 37.43)^2 + (36.92 - 37.43)^2\right]$$

$$= 50\left[(0.51)^2 + (-0.51)^2\right] = 26.01 \qquad df_A = 2 - 1 = 1$$

$$SSB(A) = c \sum_{i=1}^{2} \sum_{j=1}^{5} (\overline{Y}_{ij.} - \overline{Y}_{i..})^2$$

j	$(\overline{Y}_{1j} - \overline{Y}_{1..})^2$	$(\overline{Y}_{2j.} - \overline{Y}_{2..})^2$
1	44.36	5.66
2	1.35	1.17
3	0.07	0.05
4	8.64	5.95
5	26.42	8.07
Σ	80.84	20.90

$$SSB(A) = 10\left[80.84 + 20.90\right] = 1017.40$$

$$df_{B(A)} = 2(5-1) = 8$$

$$SSE = (c-1) \sum_i \sum_j s_{ij}^2 = 9\left[3.4^2 + 3.9^2 + 3.0^2 + 4.1^2 + 2.7^2 + 1.5^2 + 3.0^2 + 1.7^2 + 2.7^2 + 1.6^2\right] = 754.74$$

$$df_E = 2(5)(10-1) = 90$$

CAFFEINE EXAMPLE CONTINUED

ANOVA

SOURCE	df	SS	MS
BRAND	1	26.01	26.01
ESTABLISHMENT (BRAND)	8	1017.40	127.18
ERROR	90	754.74	8.39
TOTAL	99	1798.15	—

Case 1 - TREATING ESTABLISHMENTS AS RANDOM

$H_0: \alpha_1 = \alpha_2 = 0 \qquad H_A: \alpha_1 \neq \alpha_2$

T.S. $F_A = \frac{MSA}{MSAB} = \frac{26.01}{127.18} = 0.205$

RR: $F_A \geq F_{.05,1,8} = 5.32$ DON'T REJECT H_0

$H_0: \sigma^2_{b(a)} = 0 \qquad H_A: \sigma^2_{b(a)} > 0$

T.S. $F_{AB} = \frac{MSAB}{MSE} = \frac{127.18}{8.39} = 15.16$

RR: $F_{AB} \geq F_{.05,8,90} \doteq 2.06$ REJECT H_0

CASE 2 TREATING ESTABLISHMENTS AS FIXED (PLAUSIBL

$H_0: \alpha_1 = \alpha_2 = 0 \qquad H_A: \alpha_1 \neq \alpha_2$ T.S. $F_A = \frac{MSA}{MSE} = \frac{26.01}{8.39} = 3.10$

RR: $F_A \geq F_{.05,1,90} \doteq 3.96$ Don't Reject H_0, but much higher p-value

$H_0: \beta_{j(i)} = 0 \ \forall\, i,j \qquad H_A$: NOT ALL $\beta_{j(i)} = 0$

SAME TEST AS ABOVE!

CROSSED & Nested FACTORS - SPECIAL CASE: CROSSED FACTORS-FIXED
NESTED FACTORS RANDOM

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + c_{k(j)} + (\alpha\beta)_{ij} + ac_{ik(j)} + e_{ijkl}$$

$i=1,\dots,a \quad j=1,\dots,b \quad k=1,\dots,c \quad l=1,\dots,r$

EXAMPLE: A FORMULATION (REGULAR/DIET)

B BRAND (COKE/PEPSI)

C ESTABLISHMENT (NESTED UNDER BRAND BUT NOT FORMULATION)

$$SSA = bcr \sum_i (\bar{Y}_{i\cdots} - \bar{Y}_{\cdots\cdot})^2 \qquad df_A = a-1$$

$$SSB = acr \sum_j (\bar{Y}_{\cdot j\cdot\cdot} - \bar{Y}_{\cdots\cdot})^2 \qquad df_B = b-1$$

$$SSC(B) = ar \sum_j \sum_k (\bar{Y}_{\cdot jk\cdot} - \bar{Y}_{\cdot j\cdot\cdot})^2 \qquad df_{C(B)} = b(c-1)$$

$$SSAB = cr \sum_i \sum_j (\bar{Y}_{ij\cdot\cdot} - \bar{Y}_{i\cdots} - \bar{Y}_{\cdot j\cdot\cdot} + \bar{Y}_{\cdots\cdot})^2 \qquad df_{AB} = (a-1)(b-1)$$

$$SSAC(B) = r \sum_i \sum_j \sum_k (\bar{Y}_{ijk\cdot} - \bar{Y}_{ij\cdot\cdot} - \bar{Y}_{\cdot jk\cdot} + \bar{Y}_{\cdot j\cdot\cdot})^2$$

AC Interaction w/in levels of B

$$df_{AC(B)} = b(a-1)(c-1)$$

$$SSE = \sum_i \sum_j \sum_k \sum_l (Y_{ijkl} - \bar{Y}_{ijk\cdot})^2 \qquad df_E = abc(r-1)$$

$$= (r-1) \sum_i \sum_j \sum_k S^2_{ijk}$$

7.44

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FORMULATION i	BRAND j	RESTAURANT k	$\bar{y}_{i\ldots}$	$\bar{y}_{\cdot j\ldots}$	$\bar{y}_{ij\ldots}$	$\bar{y}_{\cdot jk\cdot}$	$\bar{y}_{ijk\cdot}$	$\bar{y}_{\ldots}$	s_{ijk}
1	1	1	37.43	42.15	37.94	45.80	44.6	39.18	3.4
1	1	2	37.43	42.15	37.94	41.75	39.1	39.18	3.9
1	1	3	37.43	42.15	37.94	41.65	38.2	39.18	3.0
1	1	4	37.43	42.15	37.94	41.10	35.0	39.18	4.1
1	1	5	37.43	42.15	37.94	40.45	32.8	39.18	2.7
1	2	1	37.43	36.20	36.92	36.50	39.3	39.18	1.3
1	2	2	37.43	36.20	36.92	36.10	38.0	39.18	2.9
1	2	3	37.43	36.20	36.92	38.00	36.7	39.18	3.4
1	2	4	37.43	36.20	36.92	34.10	35.5	39.18	2.5
1	2	5	37.43	36.20	36.92	36.30	35.1	39.18	1.4
2	1	1	40.92	42.15	46.36	45.80	47.0	39.18	1.5
2	1	2	40.92	42.15	46.36	41.75	44.4	39.18	3.0
2	1	3	40.92	42.15	46.36	41.65	45.1	39.18	1.7
2	1	4	40.92	42.15	46.36	41.10	47.2	39.18	2.7
2	1	5	40.92	42.15	46.36	40.45	48.1	39.18	1.6
2	2	1	40.92	36.20	35.48	36.50	33.7	39.18	1.8
2	2	2	40.92	36.20	35.48	36.10	34.2	39.18	2.0
2	2	3	40.92	36.20	35.48	38.00	39.3	39.18	1.2
2	2	4	40.92	36.20	35.48	34.10	32.7	39.18	1.7
2	2	5	40.92	36.20	35.48	36.30	37.5	39.18	2.3

$SSA = r\sum_i\sum_j\sum_k(\bar{y}_{i\ldots}-\bar{y}_{\ldots\cdot})^2 = 10(2)(5)\left[(\bar{y}_{1\ldots}-\bar{y}_{\ldots\cdot})^2+(\bar{y}_{2\ldots}-\bar{y}_{\ldots\cdot})^2\right]$

$SSB = r\sum_i\sum_j\sum_k(\bar{y}_{\cdot j\ldots}-\bar{y}_{\ldots\cdot})^2 = 10(2)(5)\left[(\bar{y}_{\cdot 1\ldots}-\bar{y}_{\ldots\cdot})^2+(\bar{y}_{\cdot 2\ldots}-\bar{y}_{\ldots\cdot})^2\right]$

$SS(AB) = r\sum_i\sum_j\sum_k(\bar{y}_{ij\ldots}-\bar{y}_{i\ldots}-\bar{y}_{\cdot j\ldots}+\bar{y}_{\ldots\cdot})^2$

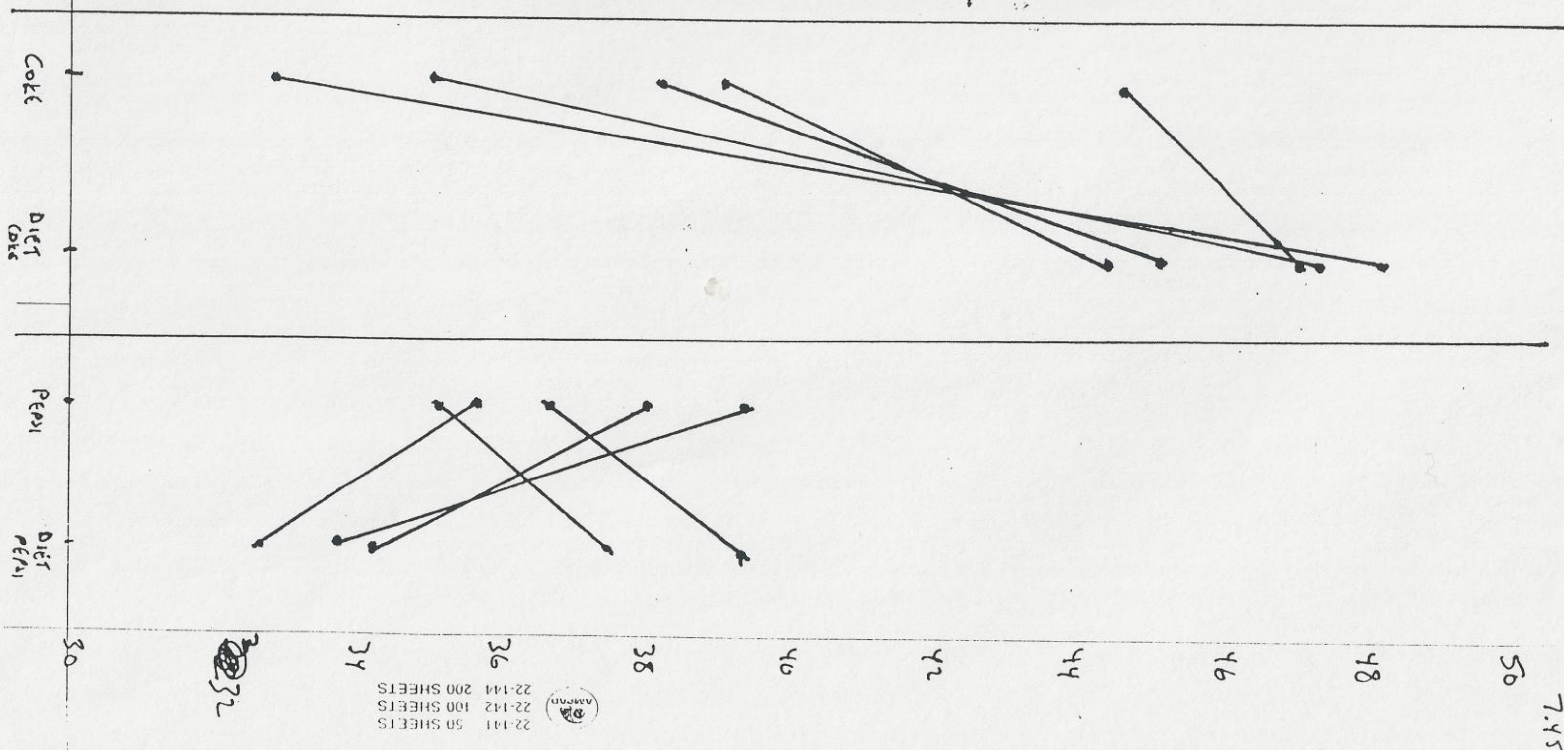
$SS(C|B) = r\sum\sum\sum(\bar{y}_{\cdot jk\cdot}-\bar{y}_{\cdot j\cdot\cdot})^2$

$SS(AC|B) = r\sum_i\sum_j\sum_k(\bar{y}_{ijk\cdot}-\bar{y}_{ij\cdot\cdot}-\bar{y}_{\cdot jk\cdot}+\bar{y}_{\cdot j\cdot\cdot})^2$

$SSE = \sum_i\sum_j\sum_k s^2_{ijk}(r_{ijk}-1)$

ANOVA

Source	df	SS	MS	E(MS)	F_0	F
A (FORM)	1	609.0	609.0	$\sigma^2 + 10\sigma^2_{ac(b)} + 100\sum\alpha_i^2$	609.0/103.6 = 5.88	5.3
B (BRAND)	1	1770.1	1770.1	$\sigma^2 + 10\sigma^2_{ac(b)} + 20\sigma^2_{c(b)} + 100\sum\beta_j^2$	1770.1/97.0 = 18.25	5.3
A×B (F×B)	1	1215.0	1215.0	$\sigma^2 + 10\sigma^2_{ac(b)} + 50\sum\sum(\alpha\beta)^2_{ij}$	1215.0/103.6 = 11.73	5.3
C|B (REST/BRAND)	8	776.2	97.0	$\sigma^2 + 10\sigma^2_{ac(b)} + 20\sigma^2_{c(b)}$	97.0/103.6 = 0.94	3.4
AC|B (REST×FORM/BRAND)	8	828.4	103.6	$\sigma^2 + 10\sigma^2_{ac(b)}$	103.6/6.5 = 15.94	≈2.
ERROR	180	1175.3	6.5	σ^2		
TOTAL	199	6374.0				



7.45

RULES FOR COMPUTING EXPECTED MEAN SQUARES

(1) Write out the appropriate Linear Model

(2) Set up a 2-WAY TABLE WHERE:

a) There is a row for each term (except μ)

b) Column for each subcript

(3) OVER EACH COLUMN, WRITE THE NUMBER OF FACTOR LEVELS & "F" (for Fixed) or "R" for Random

(4) Add column on end for components (θ^2 - fixed, σ^2 random

(5) If a column subscript does not appear in the row effect, enter the number of levels corresponding to the subscript for the column

(6) If a subscript is bracketed in a row effect, Place a 1 in cells under those subscripts that are inside brackets

(7) a) For each row, if any row subscript matches the column subscript, enter a 0 if:

i) The column represents a fixed factor

and ii) there is a fixed component of variation (θ^2) for the effect represented by the row.

b) Enter 1 in remaining cells

→

8. To determine $E[MS]$

a) Include σ^2 w/ a coefficient of 1 in all $E(MS)$

b) Of the other variance components, include only those whose model terms include subscripts of the effect of interest

c) Cover columns of non-bracketed subscripts for the effect (row) of interest

d) The coefficient for each component in in the $E[MS]$ is the product of the remaining columns of the row for that effect

EXAMPLE – FORMULATION/BRAND/ESTABLISHMENT EXAMPLE

STEP 1 $Y_{ijkl} = \mu + \alpha_i + \beta_j + c_{k(j)} + (\alpha\beta)_{ij} + ac_{ik(j)} + e_{ijkl}$

$i=1,\dots,a \quad j=1,\dots,b \quad k=1,\dots,c \quad l=1,\dots,r$

STEP 2

Source	Effect	F/R Levels Subscript: F a i	F b j	R c k	R r l	STEP 3	STEP 4 Component
A	α_i						$\theta_a^2 = \frac{\sum \alpha_i^2}{a-1}$
B	β_j						θ_b^2
AB	$(\alpha\beta)_{ij}$						θ_{ab}^2
C(B)	$c_{k(j)}$						$\sigma_{c(b)}^2$
AC(B)	$(ac)_{ik(j)}$						$\sigma_{ac(b)}^2$
ERROR	$e_{l(ijk)}$						σ_e^2

NOTE: CIRCLED VALUES OCCURRED IN CURRENT STEP

Step 5

- Row 1 (A) has only i in it, place the # of levels for each column that does not have i as subscript
- Repeat for all rows

		i	j	k	l	
A	α_i		b	c	r	σ_a^2
B	β_j	a		c	r	σ_b^2
AB	$(\alpha\beta)_{ij}$			c	r	σ_{ab}^2
C(B)	$c_{k(j)}$	a			r	$\sigma_{c(b)}^2$
AC(B)	$(ac)_{ik(j)}$				r	$\sigma_{ac(b)}^2$
Error	$e_{l(ijk)}$					σ_e^2

STEP 6

- ROW 4 (C(B)) & Row 5 (AC(B)) have j's bracketed
- Row 6 (Error) has i, j, k bracketed.

Place 1's in these cells under bracketed subscripts

	i	j	k	l	
α_i		b	c	r	σ_a^2
β_j	a		c	r	σ_b^2
$\alpha\beta_{ij}$			c	r	σ_{ab}^2
$c_{k(j)}$	a	1		r	$\sigma_{c(b)}^2$
$(ac)_{ik(j)}$		1		r	$\sigma_{ac(b)}^2$
$e_{l(ijk)}$	1	1	1		σ_e^2

STEP 7 a) • Row 1(A) has subscript i

i) Column i is FIXED (F)

ii) Row Component is Fixed (θ)

$\Rightarrow$ $\boxed{\text{Place 0 in Column } i \text{ in row 1}}$

• Row 4 ($c(B)$) has subscript K

i) Column k is RANDOM

ii) Row Component is Variance component σ^2

$\Rightarrow$ $\boxed{\text{Do not Place 0 in Column } k \text{ in row}}$

	F a i	F b j	R c k	R r l	
α_i	0	b	c	r	θ_a^2
β_j	a	0	c	r	θ_b^2
$(\alpha\beta)_{ij}$	0	0	c	r	θ_{ab}^2
$c_{k(j)}$	a	1		r	$\sigma^2_{c(b)}$
$(ac)_{ik(j)}$		1		r	$\sigma^2_{ac(b)}$
$e_{l(ijk)}$	1	1	1		σ_e^2

Note no 0 in Column i Since row Component is σ^2

b) Place 1's every where else

	i	j	k	l	
α_i	0	b	c	r	θ_a^2
β_j	a	0	c	r	θ_b^2
$(\alpha\beta)_{ij}$	0	0	c	r	θ_{ab}^2
$c_{k(j)}$	a	1	1	r	$\sigma^2_{c(b)}$
$(ac)_{ik(j)}$	1	1	1	r	$\sigma^2_{ac(b)}$
$e_{l(ikj)}$	1	1	1	1	σ_e^2

STEP 8

a) For EACH $E[MS]$, include $1\sigma^2$

b) of other Variance Components include only those whose model terms include subscripts of the effect of interest

Effect		INCLUDED VARIANCE COMPONENTS
A	α_i	$\sigma^2, \sigma^2_{ac(b)}$
B	β_j	$\sigma^2, \sigma^2_{ac(b)}, \sigma^2_{c(b)}$
AB	$(\alpha\beta)_{ij}$	$\sigma^2, \sigma^2_{ac(b)}$
C(B)	$c_{k(j)}$	$\sigma^2, \sigma^2_{c(b)}, \sigma^2_{ac(b)}$
AC(B)	$ac_{ik(j)}$	$\sigma^2, \sigma^2_{ac(b)}$
Error	$e_{l(ijk)}$	σ^2

c) Cover columns of non-bracketed subscripts for each row (1-at-a-time)

FACTOR A:
(α_i)

	i	j	k	l	Component	Product uncovered col.
α_i	~~0~~	b	c	r	θ^2_a	bcr
β_j	~~a~~	0	c	r	θ^2_b	0
$(\alpha\beta)_{ij}$	~~0~~	0	c	r	θ^2_{ab}	0
~~$c_{k(j)}$~~	~~a~~	~~1~~	~~1~~	r	~~$\sigma^2_{c(b)}$~~	r (NOT in set of VC
$ac_{ik(j)}$	~~1~~	1	1	r	$\sigma^2_{ac(b)}$	r
$e_{l(ijk)}$	~~1~~	1	1	1	σ^2_e	1

d) Obtain product of remaining columns (not covered)

$\rightarrow$ $\boxed{E[MSA] \quad bcr\,\theta^2 \quad + r\sigma^2 \ldots + \sigma_e^2}$

FACTOR C:

$C_{k(j)}$

	i	j	k	l	Component	Product
α_i	0	b	$\cancel{c}$	r	θ_a^2	0
β_j	a	0	$\cancel{c}$	r	θ_b^2	0
$(\alpha\beta)_{ij}$	0	0	$\cancel{c}$	r	θ_{ab}^2	0
$c_{k(j)}$	a	1	$\cancel{1}$	r	$\sigma_{c(b)}^2$	ar
$ac_{ik(j)}$	1	1	$\cancel{1}$	r	$\sigma_{ac(b)}^2$	r
$e_{l(ijk)}$	1	1	$\cancel{1}$	1	σ_e^2	1

$$E[MSC] = ar\sigma_{c(b)}^2 + r\sigma_{ac(b)}^2 + \sigma_e^2$$

ANALYSIS OF VARIANCE

SOURCE	df	E[MS]
A	$a-1$	$bcr\cdot\theta_a^2 + r\sigma_{ac(b)}^2 + \sigma_e^2$
B	$b-1$	$acr\,\theta_b^2 + r\sigma_{ac(b)}^2 + ar\sigma_{c(b)}^2 + \sigma_e^2$
AB	$(a-1)(b-1)$	$cr\theta_{ab}^2 + r\sigma_{ac(b)}^2 + \sigma_e^2$
C(B)	$b(c-1)$	$ar\sigma_{c(b)}^2 + r\sigma_{ac(b)}^2 + \sigma_e^2$
AC(B)	$(a-1)b(c-1)$	$r\sigma_{ac(b)}^2 + \sigma_e^2$
Error	$abc(r-1)$	σ_e^2
TOTAL	$abcr-1$	