Chapter 6 - FACTORIAL TREATMENT DESIGNS

- Can improve efficiency of Experiments by varying multiple <u>FACTORS</u> simultaneously.

- Factors ≡ Types of Treatments

EXAMPLE 1 - WEIGHT GAIN IN HIV$^+$ PATIENTS

(Klausner, et al, (1996) "The Effect of Thalidomide ..." Journal of Acquired Immune Deficiency Syndromes & Human Retrovirology, 11: 247-257)

Y ≡ 21 DAY WEIGHT GAIN IN HIV-1$^+$ PATIENTS

FACTOR A ≡ DRUG TREATMENT: Thalidomide / Placebo
FACTOR B ≡ TUBERCULOSIS INFECTION STATUS: TB$^+$/TB$^-$

<u>NOTE</u>: FACTOR A IS CONTROLLED, BUT FACTOR B OCCURS NATURALLY

EXAMPLE 2 - MARKETING EFFECTS OF KNOWLEDGE / TIME

(IYER (1989) "UNPLANNED PURCHASING: KNOWLEDGE OF SHOPPING ENVIRONMENT AND TIME PRESSURE", JOURNAL OF RETAILING, 65: 40-57)

Y ≡ # of unplanned purchases on shopping trip

FACTOR A ≡ STORE KNOWLEDGE (FAMILIAR/UNFAMILIAR)
FACTOR B ≡ TIME CONSTRAINT (PRESENT/ABSENT)

<u>NOTE</u>: EXPERIMENT CONDUCTED AS CRD IN 4 TRTS (COMBINATIONS OF FACTOR LEVELS)

GENERAL 2-FACTOR FACTORIAL STRUCTURE

FACTOR A @ a Levels

FACTOR B @ b Levels

RUN IN CRD w/ ab TRTS

FACTOR B

$$
\text{FACTOR } A \quad
\begin{array}{c|cccc|c}
 & 1 & 2 & \cdots & b & \text{ROW MEANS} \\
\hline
1 & \mu_{11} & \mu_{12} & \cdots & \mu_{1b} & \bar{\mu}_{1\cdot} \\
2 & \mu_{21} & \mu_{22} & \cdots & \mu_{2b} & \bar{\mu}_{2\cdot} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a & \mu_{a1} & \mu_{a2} & \cdots & \mu_{ab} & \bar{\mu}_{a\cdot} \\
\hline
 & \bar{\mu}_{\cdot 1} & \bar{\mu}_{\cdot 2} & \cdots & \bar{\mu}_{\cdot b} & \bar{\mu}_{\cdot\cdot}
\end{array}
$$

Means FOR LEVELS OF FACTOR A

overall mean

MEANS FOR LEVELS OF FACTOR B

① Simple Effects

Comparisons of LEVELS OF ONE FACTOR WITHIN LEVELS OF OTHER FACTOR.

$\mu_{11} - \mu_{21}$ Compares mean of Levels 1 & 2 of FACTOR A WHEN FACTOR B IS @ LEVEL 1

$\mu_{23} - \mu_{25}$ Compares MEANS OF LEVELS 3 & 5 OF FACTOR B WHEN FACTOR A IS @ LEVEL 2

(2) MAIN EFFECTS

Comparisons OF AVERAGE LEVELS OF FACTORS

$\mu_{2.} - \mu_{1.}$ COMPARES MEANS (ACROSS ALL LEVELS OF FACTOR B) OF LEVELS 1 & 2 OF FACTOR A.

$\mu_{.3} - \mu_{.5}$ COMPARES MEANS (ACROSS ALL LEVELS OF FACTOR A) OF LEVELS 3 & 5 OF FACTOR B.

(3) Interaction Effects

Comparisons of Simple Effects Across LEVELS OF THE OTHER FACTOR

$(\mu_{11} - \mu_{21}) - (\mu_{13} - \mu_{23})$ Compares simple effects of ~~[illegible]~~ levels 1 & 2 of factor A ACROSS LEVELS 1 & 3 OF FACTOR B

NOTE: $(\mu_{11} - \mu_{21}) - (\mu_{13} - \mu_{23}) =$

$= (\mu_{11} - \mu_{13}) - (\mu_{21} - \mu_{23})$

Compares simple effects of Levels 1 & 3 OF FACTOR B ACROSS LEVELS 1 & 2 OF FACTOR A.

See plots on p. 180 of Kuehl

STATISTICAL MODEL FOR 2 FACTORS

① Cell Means Model (1-WAY ANOVA with t=ab TRTS)

[FIXED FACTORS]

$$Y_{ijk} = \mu_{ij} + e_{ijk} \qquad \begin{array}{l} i = 1, \dots, a \\ j = 1, \dots, b \\ k = 1, \dots, r \end{array}$$

$\mu_{ij} \equiv$ Mean when factor A is @ level i, B is @ j

$E[e_{ijk}] = 0 \quad V[e_{ijk}] = \sigma^2$ (Need Normality & independence for F, t-test

Least Squares Estimation

$$Q = \sum_i \sum_j \sum_k (Y_{ijk} - \mu_{ij})^2 = \sum_i \sum_j \sum_k e_{ijk}^2$$

$\min_{\{\mu_{ij}\}} Q \Rightarrow \dfrac{\partial Q}{\partial \mu_{ij}} = 0$ & solve for $\hat{\mu}_{ij}$

NOTE :
$$\frac{\partial (Y_{ijk} - \mu_{ij})^2}{\partial \mu_{i'j'}} = \begin{cases} 2(Y_{ijk} - \mu_{ij})(-1) & i = i' \\ & j = j' \\ 0 & \text{o.w.} \end{cases}$$

$$\frac{\partial Q}{\partial \mu_{ij}} = \sum_k (-2)(Y_{ijk} - \mu_{ij}) + \sum_{i' \neq i} \sum_j \sum_k 0 + \sum_i \sum_{j' \neq j} \sum_k 0$$

$\rightarrow$

Setting $\frac{\partial Q}{\partial \mu_{ij}} = 0$

$$\Rightarrow -2 \sum_k (y_{ijk} - \hat{\mu}_{ij}) = 0$$

$$\Rightarrow \sum_k (y_{ijk} - \hat{\mu}_{ij}) = 0$$

$$\Rightarrow \sum_k y_{ijk} = r\hat{\mu}_{ij} \Rightarrow \boxed{\hat{\mu}_{ij} = \frac{\sum_k y_{ijk}}{r} = \bar{y}_{ij\cdot}}$$

$i = 1, \ldots, a \quad j = 1, \ldots, b$

$\hat{\bar{\mu}}_{i\cdot} = \bar{y}_{i\cdot\cdot} \qquad i = 1, \ldots, a$

$\hat{\bar{\mu}}_{\cdot j} = \bar{y}_{\cdot j\cdot} \qquad j = 1, \ldots, b$

$\hat{\bar{\mu}}_{\cdot\cdot} = \bar{y}_{\cdot\cdot\cdot}$

Additivity & MAIN FACTOR EFFECTS MODEL

(2A) MAIN EFFECTS MODEL (No Interaction)

$$\mu_{ij} = \underbrace{\bar{\mu}_{\cdot\cdot}}_{\text{Overall mean}} + \underbrace{(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})}_{\text{Level } i \text{ of factor A main effect}} + \underbrace{(\bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot\cdot})}_{\text{Level } j \text{ of Factor B main Effects}}$$

Additive Effects

$$\mu_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j$$

$\alpha_i = \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot}$

$\beta_j = \bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot\cdot}$

2B INTERACTION MODEL

EFFECTS OF LEVELS OF FACTOR A DIFFER AMONG LEVELS OF FACTOR B & VICEVERSA

Interaction Effect of LEVEL i OF FACTOR A & LEVEL j OF FACTOR B IS

Difference between TRT Effect of Combination of A @ i, and B @ j and Corresponding Main Effects:

$\mu_{ij} - \bar{\mu}_{..}$ ≡ TRT Effect of combination of A @ Level i & B @ Level j

$\bar{\mu}_{i.} - \bar{\mu}_{..}$ ≡ MAIN Effect of A @ Level i

$\bar{\mu}_{.j} - \bar{\mu}_{..}$ ≡ Main Effect of B @ Level j

$$(\mu_{ij} - \bar{\mu}_{..}) - \left\{ \left[\bar{\mu}_{i.} - \bar{\mu}_{..} \right] + \left[\bar{\mu}_{.j} - \bar{\mu}_{..} \right] \right\}$$

$\equiv (\alpha\beta)_{ij}$ ≡ Interaction Effect between Factor A @ i and Factor B @ j

$$\alpha\beta_{ij} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$$

Effects Model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad \begin{array}{l} i = 1, \dots, a \\ j = 1, \dots, b \\ k = 1, \dots, r \end{array}$$

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$$

By their definitions above

Alternative Model (Solution provided by SAS' PROC GLM)

$$\mu^* = \mu + \alpha_a + \beta_b + (\alpha\beta)_{ab}$$

$$\alpha_i^* = (\alpha_i - \alpha_a) + \left[(\alpha\beta)_{ib} - (\alpha\beta)_{ab}\right] \qquad i = 1, \dots, a-1$$

$$\beta_j^* = (\beta_j - \beta_b) + \left[(\alpha\beta)_{aj} - (\alpha\beta)_{ab}\right] \qquad j = 1, \dots, b-1$$

$$(\alpha\beta)_{ij}^* = (\alpha\beta)_{ij} - (\alpha\beta)_{ib} - (\alpha\beta)_{aj} + (\alpha\beta)_{ab}$$

$$\alpha_a = \beta_b = \alpha\beta_{ib} = \alpha\beta_{aj} = 0 \qquad \forall\, i, j$$

Any <u>estimable</u> <u>function</u> is identical under the 2 formulations

Sums of squares partition

$$\underbrace{y_{ijk} - \bar{y}_{...}} = (y_{ijk} - \bar{y}_{ij.}) + (\bar{y}_{ij.} - \bar{y}_{...})$$

"TOTAL DEVIATION FROM OVERALL MEAN"   "EXPERIMENTAL ERROR"   "TREATMENT EFFECT"

$$\sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{...})^2 = \underbrace{\sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})^2}_{A}$$

$$+ \underbrace{\sum_i \sum_j \sum_k (\bar{y}_{ij.} - \bar{y}_{...})^2}_{B} + \underbrace{2\sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})(\bar{y}_{ij.} - \bar{y}_{...})}_{C}$$

Consider C

$$2\sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})(\bar{y}_{ij.} - \bar{y}_{...}) = 2\sum_i \sum_j \cancelto{0}{(\bar{y}_{ij.} - \bar{y}_{...})} \sum_k \cancelto{0}{(y_{ijk} - \bar{y}_{ij.})}$$

$$\Rightarrow SSTOTAL = \underbrace{SS\,ERROR}_{A} + \underbrace{SSTREATMENT}_{B}$$

$SSTOTAL = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{...})^2$ $df_{TOTAL} = abr - 1$

$SSERROR = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})^2 = (r-1) \sum_i \sum_j S_{ij}^2$ $df_{ERR} = ab(r-$

$SS_{TRTS} = r \sum_i \sum_j (\bar{y}_{ij.} - \bar{y}_{...})^2$ $df_{TRTS} = (a-1)(b-1) + (a-1) + (b-$

SS TREATMENTS CAN BE DECOMPOSED INTO <u>MAIN EFFECTS</u> FOR FACTORS A & B AND THE INTERACTION BETWEEN LEVELS OF FACTORS OF A & B

$$(\bar{y}_{ij.}-\bar{y}_{...}) = (\bar{y}_{i..}-\bar{y}_{...}) + (\bar{y}_{.j.}-\bar{y}_{...}) + (\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})$$

$$\sum_i\sum_j\sum_k(\bar{y}_{ij.}-\bar{y}_{...})^2 = rb\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2 + ra\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2$$
$$+ r\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$$

All cross product terms drop out

$$SS_{TRTS} = SSA + SSB + SSAB$$

<u>where</u> $SSA = rb\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2 \quad df_A = a-1$

$SSB = ra\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2 \quad df_B = b-1$

$SSAB = r\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2 \quad df_{AB} = (a-1)(b$

Analysis of Variance ([redacted]

Source	df	SS	MS	F_0
Factor A	$a-1$	SSA	$MSA = \frac{SSA}{a-1}$	$F_A = MSA/MSE$
Factor B	$b-1$	SSB	$MSB = \frac{SSB}{b-1}$	$F_B = MSB/MSE$
Interaction	$(a-1)(b-1)$	SSAB	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$F_{AB} = \frac{MSAB}{MSE}$
Error	$ab(r-1)$	SSE	$MSE = \frac{SSE}{ab(r-1)}$	
TOTAL	$abr-1$	SSTOTAL	—	

Procedure:

(1) Test for interaction: If interaction is significant, treatment effects exist and main effects tests may not be appropriate.

(2) If interaction is not significant, test for main effects for factors A & B.

Test for Interaction

$H_0: (\alpha\beta)_{11} = \cdots = (\alpha\beta)_{ab} = 0$ (No interaction)

$H_A:$ Not all $(\alpha\beta)_{ij} = 0$ (Interaction exists)

T.S. $F_{AB} = \frac{MSAB}{MSE}$

RR: $F_{AB} \geq F_{\alpha,(a-1)(b-1),\, ab(r-1)}$

Test for Main Effects (Factor A)

$H_0: \alpha_1 = \cdots = \alpha_a = 0$ (No factor A main effects)

H_A: Not all $\alpha_i = 0$ (Factor A main Effects exist)

T.S. $F_A = \dfrac{MSA}{MSE}$

RR: $F_A \geq F_{\alpha, a-1, ab(r-1)}$

obvious adjustments to test for factor B effects.

Least Squares Estimation for Factorial Trt Designs (Equal r

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}$$

$$Q = \sum_i \sum_j \sum_k e_{ijk}^2 = \sum_i \sum_j \sum_k \left(y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right)^2$$

$$\frac{\partial Q}{\partial \mu} = 2 \sum_i \sum_j \sum_k \left(y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \widehat{(\alpha\beta)}_{ij}\right)(-1) = 0$$

$$\frac{\partial Q}{\partial \alpha_i} = 2 \sum_j \sum_k \left(y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \widehat{(\alpha\beta)}_{ij}\right)(-1) = 0 \quad i = 1, \ldots, a$$

$$\frac{\partial Q}{\partial \beta_j} = 2 \sum_i \sum_k \left(y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \widehat{(\alpha\beta)}_{ij}\right)(-1) = 0 \quad j = 1, \ldots, b$$

$$\frac{\partial Q}{\partial (\alpha\beta)_{ij}} = 2 \sum_k \left(y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \widehat{(\alpha\beta)}_{ij}\right)(-1) = 0 \quad \begin{array}{l} i = 1, \ldots, \\ j = 1, \ldots, \end{array}$$

$\Rightarrow$

$$\mu: \quad \sum_i\sum_j\sum_k y_{ijk} = abr\hat{\mu} + br\sum_i \hat{\alpha}_i + ar\sum_j \hat{\beta}_j + r\sum_i\sum_j \widehat{(\alpha\beta)}_{ij}$$

$$\alpha_i: \quad \sum_j\sum_k y_{ijk} = br\hat{\mu} + br\,\hat{\alpha}_i + r\sum_j \hat{\beta}_j + r\sum_j \widehat{(\alpha\beta)}_{ij} \qquad i=1,\dots,a$$

$$\beta_j: \quad \sum_i\sum_k y_{ijk} = ar\hat{\mu} + r\sum_i \hat{\alpha}_i + ar\hat{\beta}_j + r\sum_i \widehat{(\alpha\beta)}_{ij} \qquad j=1,\dots,b$$

$$(\alpha\beta)_{ij}: \quad \sum_k y_{ijk} = r\hat{\mu} + r\hat{\alpha}_i + r\hat{\beta}_j + r\widehat{(\alpha\beta)}_{ij}$$

Note that the set of $\{\alpha_i\}$ equations sum to the μ equation, similarly for the sets of $\{\beta_j\}$ and $\{(\alpha\beta)_{ij}\}$ equations.

Using constraints $\sum_i \hat{\alpha}_i = \sum_j \hat{\beta}_j = \sum_i \widehat{(\alpha\beta)}_{ij} = \sum_j \widehat{(\alpha\beta)}_{ij} = 0$
give ~~wai~~ a set of unique solutions:

$$\mu: \quad y_{\cdot\cdot\cdot} = abr\hat{\mu} \Rightarrow \hat{\mu} = \frac{y_{\cdot\cdot\cdot}}{abr} = \bar{y}_{\cdot\cdot\cdot}$$

$$\alpha_i: \quad y_{i\cdot\cdot} = br\hat{\mu} + br\hat{\alpha}_i \Rightarrow \hat{\mu} + \hat{\alpha}_i = \frac{y_{i\cdot\cdot}}{br} = \bar{y}_{i\cdot\cdot}$$

$$\Rightarrow \hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}$$

β_j : $y_{\cdot j \cdot} = ar\hat{\mu} + ar\hat{\beta}_j \Rightarrow \bar{y}_{\cdot j \cdot} = \hat{\mu} + \hat{\beta}_j$

$\Rightarrow \hat{\beta}_j = \bar{y}_{\cdot j \cdot} - \bar{y}_{\cdot\cdot\cdot}$

$(\alpha\beta)_{ij}$: $y_{ij\cdot} = r\hat{\mu} + r\hat{\alpha}_i + r\hat{\beta}_j + r\widehat{(\alpha\beta)}_{ij}$

$\Rightarrow \bar{y}_{ij\cdot} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \widehat{(\alpha\beta)}_{ij}$

$\Rightarrow \widehat{(\alpha\beta)}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{\cdot\cdot\cdot} - (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}) - (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})$

$= \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot}$

<u>NOTE:</u> $SSA = br \sum_i \hat{\alpha}_i^2$

$SSB = ar \sum_j \hat{\beta}_j^2$

$SSAB = r \sum_i \sum_j \widehat{(\alpha\beta)}_{ij}^2$

EXPECTED MEAN SQUARES (Fixed Effects Model)

$SSA = br \sum_i (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 = br \sum_i \bar{y}_{i\cdot\cdot}^2 - abr\,\bar{y}_{\cdot\cdot\cdot}^2$

$SSAB = r \sum_i \sum_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2$

$SSE = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij\cdot})^2 = \sum_i \sum_j \sum_k y_{ijk}^2 - r \sum_i \sum_j \bar{y}_{ij\cdot}^2$

EXPANDING SSAB

$$r\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$$

$$= r\sum_i\sum_j \bar{y}_{ij.}^2 + rb\sum_i \bar{y}_{i..}^2 + ra\sum_j \bar{y}_{.j.}^2 + rab\bar{y}_{...}^2$$

$$+ \overset{A}{2r\bar{y}_{...}\sum_i\sum_j \bar{y}_{ij.}} - \overset{B}{2r\sum_i \bar{y}_{i..}\sum_j \bar{y}_{ij.}} - \overset{C}{2r\sum_j \bar{y}_{.j.}\sum_i \bar{y}_{ij.}}$$

$$+ \overset{D}{2r\sum_i \bar{y}_{i..}\sum_j \bar{y}_{.j.}} - \overset{E}{2br\bar{y}_{...}\sum_i \bar{y}_{i..}}$$

$$- \underset{F}{2ar\bar{y}_{...}\sum_j \bar{y}_{.j.}}$$

A $\quad 2r\bar{y}_{...}\sum_i\sum_j \bar{y}_{ij.} = 2r\bar{y}_{...}\, ab\bar{y}_{...} = 2abr\bar{y}_{...}^2$

B $\quad 2r\sum_i \bar{y}_{i..}\sum_j \bar{y}_{ij.} = 2br\sum_i \bar{y}_{i..}^2$

C $\quad 2r\sum_j \bar{y}_{.j}\sum_i \bar{y}_{ij.} = 2ar\sum_j \bar{y}_{.j.}^2$

D $\quad 2r\sum_i \bar{y}_{i..}\sum_j \bar{y}_{.j.} = 2abr\bar{y}_{...}^2$

E $\quad 2br\bar{y}_{...}\sum_i \bar{y}_{i..} = 2abr\bar{y}_{...}^2$

F $\quad 2ar\bar{y}_{...}\sum_j \bar{y}_{.j.} = 2abr\bar{y}_{...}^2$

A $-$ B $-$ C $+$ D $-$ E $-$ F $= -2br\sum_i \bar{y}_{i..}^2 - 2ar\sum_j \bar{y}_{.j.}^2$

$$\Rightarrow SSAB = r\sum_i\sum_j \bar{y}_{ij.}^2 - br\sum_i \bar{y}_{i..}^2 - ar\sum_j \bar{y}_{.j.}^2 + rab\bar{y}_{...}^2$$

$$E[Y_{ijk}] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} \qquad V[Y_{ijk}] = \sigma^2$$

$$E[Y_{ijk}^2] = \sigma^2 + (\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})^2$$

$$E[\bar{Y}_{ij.}] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} \qquad V[\bar{Y}_{ij.}] = \frac{\sigma^2}{r}$$

$$E[\bar{Y}_{ij.}^2] = \frac{\sigma^2}{r} + (\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij})^2$$

$$E[\bar{Y}_{i..}] = \mu + \alpha_i \qquad V[\bar{Y}_{i..}] = \frac{\sigma^2}{br}$$

$$E[\bar{Y}_{i..}^2] = \frac{\sigma^2}{br} + (\mu + \alpha_i)^2$$

$$E[\bar{Y}_{.j.}] = \mu + \beta_j \qquad V[\bar{Y}_{.j.}] = \frac{\sigma^2}{ar}$$

$$E[\bar{Y}_{.j.}^2] = \frac{\sigma^2}{ar} + (\mu + \beta_j)^2$$

$$E[\bar{Y}_{...}] = \mu \qquad V[\bar{Y}_{...}] = \frac{\sigma^2}{abr}$$

$$E[\bar{Y}_{...}^2] = \frac{\sigma^2}{abr} + \mu^2$$

$$E[SSA] = br \sum_i E[\bar{y}_{i..}^2] - abr E[\bar{y}_{...}^2]$$

$$= br \sum_{i=1}^{a} \left[\frac{\sigma^2}{br} + (\mu+\alpha_i)^2\right] - abr\left[\frac{\sigma^2}{abr} + \mu^2\right]$$

$$= a\sigma^2 + br \sum_i (\mu^2 + 2\mu\alpha_i + \alpha_i^2) - \sigma^2 - abr\mu^2$$

$$= a\sigma^2 + \cancel{abr\mu^2} + 2\mu br \cancel{\sum \alpha_i}^{\,0} + br\sum \alpha_i^2 - \sigma^2 - \cancel{abr\mu^2}$$

$$= (a-1)\sigma^2 + br \sum_i \alpha_i^2$$

$$\Rightarrow \boxed{E[MSA] = E\left[\frac{SSA}{a-1}\right] = \sigma^2 + \frac{br\sum_i \alpha_i^2}{a-1}}$$

$$E[SSAB] = r\sum_i\sum_j E[\bar{y}_{ij.}^2] - br\sum_i E[\bar{y}_{i..}^2] - ar\sum_j E(\bar{y}_{.j.}^2) + rabE(\bar{y}_{...}^2$$

$$= r\sum_i\sum_j\left\{\frac{\sigma^2}{r} + [\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}]^2\right\} - br\sum_i\left\{\frac{\sigma^2}{br} + (\mu+\alpha_i)^2\right\}$$

$$- ar\sum_j\left\{\frac{\sigma^2}{ar} + (\mu+\beta_j)^2\right\} + rab\left\{\frac{\sigma^2}{abr} + \mu^2\right\}$$

$$= ab\sigma^2 + r\sum_i\sum_j\left[\mu^2 + \alpha_i^2 + \beta_j^2 + (\alpha\beta)_{ij}^2 + 2\mu\alpha_i + 2\mu\beta_j\right.$$

$$\left. + 2\mu(\alpha\beta)_{ij} + 2\alpha_i\beta_j + 2\alpha_i(\alpha\beta)_{ij} + 2\beta_j(\alpha\beta_{ij})\right]$$

$$- a\sigma^2 - br\sum_i[\mu^2 + \alpha_i^2 + 2\mu\alpha_i] - b\sigma^2 - ar\sum_j[\mu^2 + \beta_j^2 + 2\mu\beta$$

$$+ \sigma^2 + rab\mu^2$$

$$= ab\sigma^2 + r\cancel{ab}\mu^2 + rb\sum_i \cancel{\alpha_i}^2 + ra\sum_j \beta_j^2 + r\sum_i\sum_j (\alpha\beta)_{ij}^2$$

$$+ 2r\mu b \cancel{\sum_i \alpha_i} + 2r\mu a \sum_j \cancel{\beta_j} + 2r\mu \sum_i\sum_j \cancel{(\alpha\beta)_{ij}}$$

$$+ 2r \sum_i \cancel{\alpha_i} \sum_j \cancel{\beta_j} + \cancel{2r\sum_j (\alpha\beta)_{ij} \sum_i} \; 2r \sum_i \alpha_i \sum_j \cancel{(\alpha\beta)_{ij}}$$

$$+ 2r \sum_j \beta_j \sum_i \cancel{(\alpha\beta)_{ij}} - a\sigma^2 - ab\cancel{r}\mu^2 - br\sum_i \cancel{\alpha_i}^2$$

$$- 2\mu br \sum_i \cancel{\alpha_i} - b\sigma^2 - a\cancel{b}\mu^2 - ar\sum_j \cancel{\beta_j}^2$$

$$- 2ar\mu \sum_j \cancel{\beta_j} + \sigma^2 + r\cancel{ab}\mu^2$$

$$= \sigma^2 [ab - a - b + 1] + r\sum_i\sum_j (\alpha\beta)_{ij}^2$$

$$= \sigma^2 (a-1)(b-1) + r\sum_i\sum_j (\alpha\beta)_{ij}^2$$

$$\Rightarrow \boxed{E[MSAB] = E\left[\frac{SSAB}{(a-1)(b-1)}\right] = \sigma^2 + \frac{r\sum_i\sum_j (\alpha\beta)_{ij}^2}{(a-1)(b-1)}}$$

$$E[SSE] = \sum_i\sum_j\sum_k E[Y_{ijk}^2] - r\sum_i\sum_j E[\bar{Y}_{ij\cdot}^2]$$

$$= \sum_i\sum_j\sum_k \left\{ \sigma^2 + [\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}]^2 \right\}$$

$$- r\sum_i\sum_j \left\{ \frac{\sigma^2}{r} + [\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}]^2 \right\} \quad \rightarrow$$

$$= abr\sigma^2 + r\sum_i\sum_j\left(\cancel{\mu} + \alpha_i + \beta_j + (\alpha\beta)_{ij}\right)^2$$

$$- ab\sigma^2 - r\sum_i\sum_j\left(\cancel{\mu + \alpha_i} + \beta_j + (\alpha\beta)_{ij}\right)^2$$

$$= ab(r-1)\sigma^2$$

$$\Rightarrow \boxed{E[MSE] = E\left[\frac{SSE}{ab(r-1)}\right] = \sigma^2}$$

ANOVA

Source	df	SS	E[MS]
A	$a-1$	SSA	$\sigma^2 + \frac{br\sum_i \alpha_i^2}{a-1}$
B	$b-1$	SSB	$\sigma^2 + \frac{ar\sum_j \beta_j^2}{b-1}$
AB	$(a-1)(b-1)$	SSAB	$\sigma^2 + \frac{r\sum_i\sum_j (\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Error	$ab(r-1)$	SSE	σ^2
Total	$abr-1$	SSTOTAL	—

EXAMPLE 1 - WT GAIN IN HIV$^+$ PATIENTS

FACTOR A: DRUG TRT: Thalidomide / Placebo

FACTOR B: TB INFECTION STATUS: TB$^+$/TB$^-$

K	$i=1/j=1$ Thal/TB$^+$	$i=1/j=2$ Thal/TB$^-$	$i=2/j=1$ Placebo/TB$^+$	$i=2/j=2$ Placebo/TB$^-$
1	9.0	2.5	0.0	-0.5
2	6.0	3.5	1.0	0.0
3	4.5	4.0	-1.0	2.5
4	2.0	1.0	-2.0	0.5
5	2.5	0.5	-3.0	-1.5
6	3.0	4.0	-3.0	0.0
7	1.0	1.5	0.5	1.0
8	1.5	2.0	-2.5	3.5
$\bar{Y}_{ij.}$	3.6875	2.3750	-1.2500	0.6875
S_{ij}	2.6984	1.3562	1.6036	1.6243

		$j=1$ TB$^+$	$j=2$ TB$^-$
$i=1$	Thal	3.6875 (2.6984)	2.3750 (1.3562)
$i=2$	Placebo	-1.2500 (1.6036)	0.6875 (1.6243)
		$\bar{Y}_{.1.} =$ 1.21875	$\bar{Y}_{.2.} =$ 1.53125

$$\bar{Y}_{1..} = \frac{3.6875 + 2.3750}{2} = 3.03125$$

$$\bar{Y}_{2..} = \frac{-1.2500 + 0.6875}{2} = -0.28125$$

$$\bar{Y}_{...} = 1.3750$$

WT GAIN IN HIV+ PATIENTS (6.20)

$$SSA = rb \sum_i (\bar{Y}_{i..} - \bar{Y}_{...})^2$$

$$= 8(2)\left[(3.03125 - 1.3750)^2 + (-0.28125 - 1.3750)\right]^2$$

$$= 16\left[2.7432 + 2.7432\right] = 87.7824$$

$$SSB = ra \sum_j (\bar{Y}_{.j.} - \bar{Y}_{...})^2$$

$$= 8(2)\left[(1.21875 - 1.3750)^2 + (1.53125 - 1.3750)^2\right]$$

$$= 16\left[0.0244 + 0.0244\right] = 0.7808$$

$$SSAB = r \sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$$

$$= 8\left[(3.6875 - 3.03125 - 1.21875 + 1.3750)^2 + (2.3750 - 3.03125 - 1.53125 + 1.3750)^2 + (-1.2500 - (-0.28125) - 1.21875 + 1.3750)^2 + (0.6875 - (-0.28125) - 1.53125 + 1.3750)^2\right]$$

$$= 8\left[.6602 + .6602 + .6602 + .6602\right] = 21.1264$$

$$SSE = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2 = r-1 \sum_i \sum_j S_{ij}^2$$

$$= 7\left[(2.6984)^2 + (1.3562)^2 + (1.6036)^2 + (1.6243)^2\right] = 100.3137$$

ANOVA (WT GAIN IN HIV+ PATIENTS)

SOURCE	df	SS	MS	F	F_α
A (DRUG)	$2-1=1$	87.7824	87.7824	24.50	4.20
B (TB)	$2-1=1$	0.7808	0.7808	0.22	4.2
AB	$(2-1)(2-1)=1$	21.1264	21.1264	5.90	4.20
Error	$2(2)(8-1)=28$	100.3137	3.5826	—	
TOTAL	$2(2)(8)-1=31$	210.0033	—		

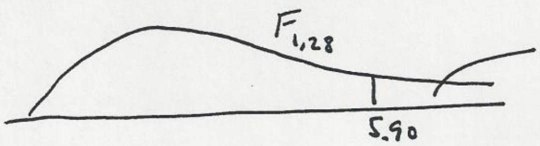
① Testing for interaction

$H_0: (\alpha\beta)_{ij}=0 \quad i=1,2 \quad j=1,2$ (No interaction)

H_A: Not all $(\alpha\beta)_{ij}=0$ (Interaction exists)

T.S. $F_{AB} = \dfrac{MSAB}{MSE} = \dfrac{21.1264}{3.5826} = 5.90$

RR: $F_{\alpha\beta} \geq F_{.05,\,1,28} = 4.20$



P_VALUE $\in (.01, .05)$

② Tests for main Effects (Not particularly meaningful since interaction exists)

$H_0: \alpha_1 = \alpha_2 = 0$ (No Drug Effect)

$H_A: \alpha_1, \alpha_2 \neq 0$ (Drug Effect)

T.S. $F_A = \dfrac{MSA}{MSE} = 24.50$

RR: $F_A \geq 4.20$ Reject H_0

$H_0: \beta_1 = \beta_2 = 0$ (No TB Effect)

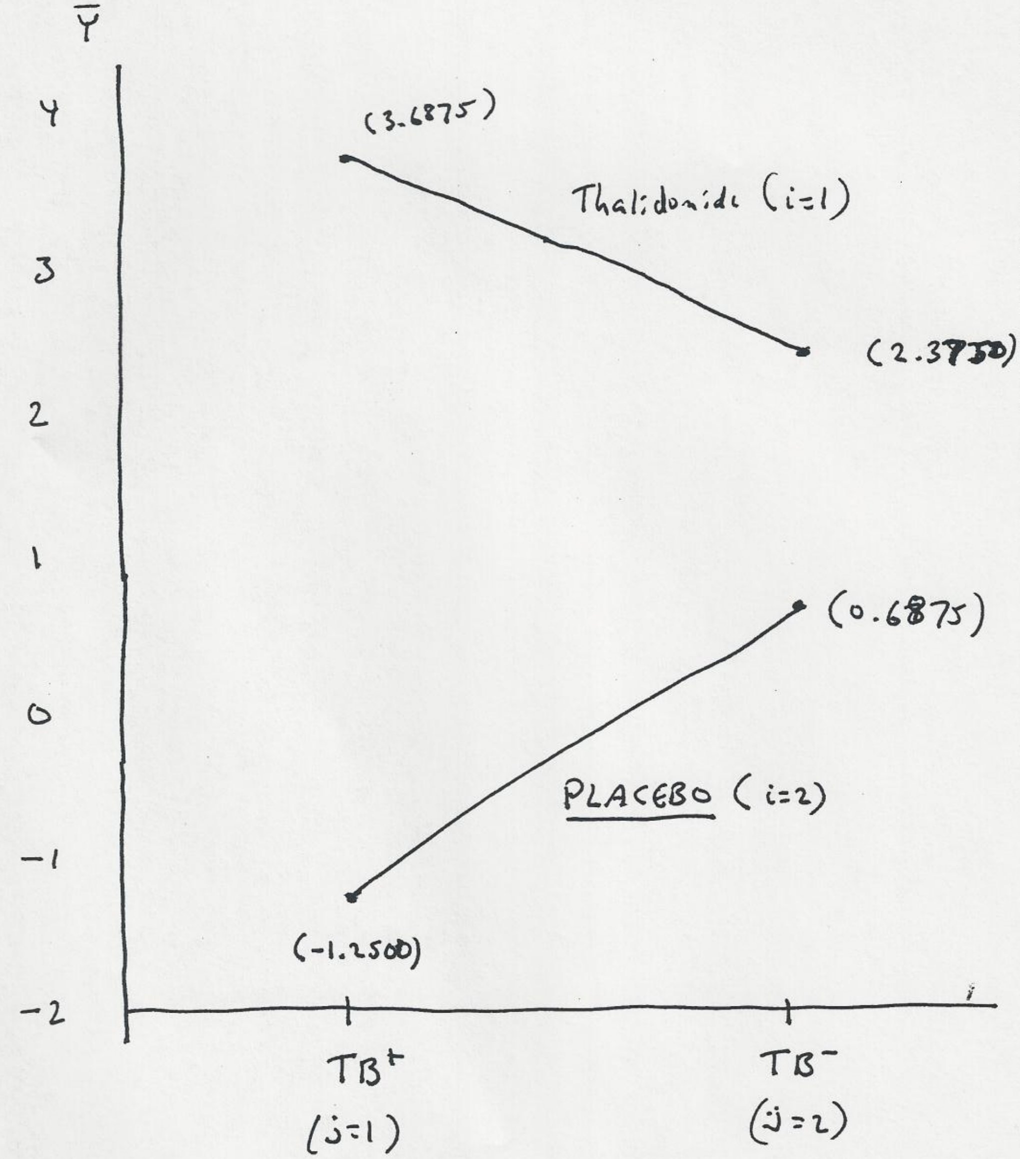
$H_A: \beta_1, \beta_2 \neq 0$ (TB Effect)

T.S. $F_B = \dfrac{MSB}{MSE} = 0.22$

RR: $F_B \geq 4.20$

DO NOT REJECT H_0.

Interaction Plot (WT GAIN IN HIV)



NOTES: ① Strong Interaction (LINES NOT PARALLEL)

② Clear Thalidomide Effect (Curves lies much above Placebo

③ No marginal TB Effect (Averages balance out)

However: Conditional on drug, large TB Effect

Standard Errors and Interval Estimates for means

Marginal means: $\bar{Y}_{i..}$, $\bar{Y}_{.j.}$

$$S^2_{\bar{Y}_{i..}} = \frac{s^2}{rb} \Rightarrow S_{\bar{Y}_{i..}} = \sqrt{\frac{MSE}{rb}}$$

$$S^2_{\bar{Y}_{.j.}} = \frac{s^2}{ra} \Rightarrow S_{\bar{Y}_{.j.}} = \sqrt{\frac{MSE}{ra}}$$

Cell means: $\bar{Y}_{ij.}$

$$S^2_{\bar{Y}_{ij.}} = \frac{s^2}{r} \Rightarrow S_{\bar{Y}_{ij.}} = \sqrt{\frac{MSE}{r}}$$

Contrasts Among MAIN EFFECTS

$$C^A = \mu_{i.} - \mu_{i'} \qquad \hat{C}^A = \bar{Y}_{i..} - \bar{Y}_{i'..}$$

$$S^2_{\hat{C}^A} = s^2\left[\frac{2}{rb}\right] \qquad S_{\hat{C}^A} = \sqrt{\frac{2\,MSE}{rb}}$$

$$C^B = \mu_{.j} - \mu_{.j'} \qquad \hat{C}^B = \bar{Y}_{.j.} - \bar{Y}_{.j'.}$$

$$S^2_{\hat{C}^B} = s^2\left[\frac{2}{ra}\right] \qquad S_{\hat{C}^B} = \sqrt{\frac{2\,MSE}{ra}}$$

Contrasts among simple effects (appropriate when interaction exists)

$$C^s = \mu_{ij} - \mu_{ij'} \quad \text{(SAY)}$$

$$c^s = \bar{Y}_{ij\cdot} - \bar{Y}_{ij'\cdot}$$

$$s^2_{c^s} = s^2\left(\frac{2}{r}\right) \qquad s_{c^s} = \sqrt{\frac{2\,MSE}{r}}$$

EXAMPLE - Simultaneous 95% CI for Thalidomide Effects in TB^+ and TB^- PATIENTS SEPARATELY

$$C^{TB^+} = \mu_{11} - \mu_{21} \qquad c^{TB^+} = \bar{Y}_{11\cdot} - \bar{Y}_{21\cdot} = 3.6875 - (-1.2500) = 4.9375$$

$$s_{c^{TB^+}} = \sqrt{\frac{MSE(2)}{r}} = \sqrt{\frac{3.5826(2)}{8}} = .9464$$

$$C^{TB^-} = \mu_{12} - \mu_{22} \qquad c^{TB^-} = \bar{Y}_{12\cdot} - \bar{Y}_{22\cdot} = 2.3750 - 0.6875 = 1.6875$$

$$s_{c^{TB^-}} = .9464$$

Bonferroni's t ($k=2$ comparisons) $t_{.025,2,28} \doteq 2.37$

TB^+: $4.9375 \pm 2.37(.9464) \equiv 4.9375 \pm 2.2430 \equiv (2.70, 7.18)$

TB^-: $1.6875 \pm 2.2430 \equiv (-0.56, 3.93)$

Large Thalidomide (vs Placebo) effect in TB^+. Small nonsignificant in TB^-

Three Treatment Factors

Cell Means Model

$$Y_{ijkl} = \mu_{ijk} + e_{ijkl} \qquad \begin{array}{l} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,c \\ l=1,\dots,r \end{array}$$

Factor Effects Model

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}$$

$$\alpha_i = \bar{\mu}_{i\cdot\cdot} - \bar{\mu}_{\cdot\cdot\cdot} \qquad \beta_j = \bar{\mu}_{\cdot j \cdot} - \bar{\mu}_{\cdot\cdot\cdot} \qquad \gamma_k = \bar{\mu}_{\cdot\cdot k} - \bar{\mu}_{\cdot\cdot\cdot}$$

$$(\alpha\beta)_{ij} = \bar{\mu}_{ij\cdot} - \bar{\mu}_{i\cdot\cdot} - \bar{\mu}_{\cdot j\cdot} + \bar{\mu}_{\cdot\cdot\cdot}$$

$$(\alpha\gamma)_{ik} = \bar{\mu}_{i\cdot k} - \bar{\mu}_{i\cdot\cdot} - \bar{\mu}_{\cdot\cdot k} + \bar{\mu}_{\cdot\cdot\cdot}$$

$$(\beta\gamma)_{jk} = \bar{\mu}_{\cdot jk} - \bar{\mu}_{\cdot j\cdot} - \bar{\mu}_{\cdot\cdot k} + \bar{\mu}_{\cdot\cdot\cdot}$$

$$(\alpha\beta\gamma)_{ijk} = \mu_{ijk} - \bar{\mu}_{ij\cdot} - \bar{\mu}_{i\cdot k} - \bar{\mu}_{\cdot jk} + \bar{\mu}_{i\cdot\cdot} + \bar{\mu}_{\cdot j\cdot} + \bar{\mu}_{\cdot\cdot k} - \bar{\mu}_{\cdot\cdot\cdot}$$

<u>Trick</u>: For any effect, start w/ mean containing current effects subscripts. Delete one subscript at a time (obtain mean with deleted subscript), multiply each mean by (-1). Then delete 2 subscripts and multiply mean by $(-1)^2$ ... keep going each time multiplying mean by $(-1)^{\#\text{deleted subscripts}}$

Sums of squares can be obtained by:

1. Replacing μ's w/ corresponding $\bar{Y}$'s
2. Squaring resulting "contrast" se
3. Summing over all data

$$SSA = \sum_i \sum_j \sum_k \sum_l (\bar{Y}_{i\ldots} - \bar{Y}_{\ldots\ldots})^2 = rbc \sum_i (\bar{Y}_{i\ldots} - \bar{Y}_{\ldots\ldots})^2 \quad df_A = a-$$

$$SSB = rac \sum_j (\bar{Y}_{.j..} - \bar{Y}_{\ldots\ldots})^2 \qquad df_B = b-1$$

$$SSC = rab \sum_k (\bar{Y}_{..k.} - \bar{Y}_{\ldots\ldots})^2 \qquad df_C = c-1$$

$$SSAB = rc \sum_i \sum_j (\bar{Y}_{ij..} - \bar{Y}_{i\ldots} - \bar{Y}_{.j..} + \bar{Y}_{\ldots\ldots})^2 \quad df_{AB} = (a-1)(b-1)$$

$$SSAC = rb \sum_i \sum_k (\bar{Y}_{i.k.} - \bar{Y}_{i\ldots} - \bar{Y}_{..k.} + \bar{Y}_{\ldots\ldots})^2 \quad df_{AC} = (a-1)(c-1)$$

$$SSBC = ra \sum_j \sum_k (\bar{Y}_{.jk.} - \bar{Y}_{.j..} - \bar{Y}_{..k.} + \bar{Y}_{\ldots\ldots})^2 \quad df_{BC} = (b-1)(c-1)$$

$$SSABC = r \sum_i \sum_j \sum_k (\bar{Y}_{ijk.} - \bar{Y}_{ij..} - \bar{Y}_{i.k.} - \bar{Y}_{.jk.} + \bar{Y}_{i\ldots} + \bar{Y}_{.j..} + \bar{Y}_{..k.} - \bar{Y}_{\ldots\ldots})^2$$

$$df_{ABC} = (a-1)(b-1)(c-1)$$

$$SSE = \sum_i \sum_j \sum_k \sum_l (Y_{ijkl} - \bar{Y}_{ijk.})^2 = (r-1) \sum_i \sum_j \sum_k S_{ijk}^2$$

$$df_E = abc(r-1)$$

$$SSTOTAL = \sum_i \sum_j \sum_k \sum_l (Y_{ijkl} - \bar{Y}_{\ldots\ldots})^2 \qquad df_{TOTAL} = abcr-1$$

ESTIMATION OF ERROR VARIANCE w/ ONE REPLICATION

- 2 QUALITATIVE FACTORS

Problem: With No replication (beyond $r=1$ observation per cell), you cannot obtain an estimate of experimental error ($S_{ij}=0 \;\forall\, i,j$)

A Solution (Tukey) 1 Degree of Freedom Test for non-addi

Instead of having $(a-1)(b-1)$ degrees of freedom for interaction $(\alpha\beta)_{ij}$ $\quad i=1,\ldots,a$; $j=1,\ldots,b$

Assume that interaction has a specific multiplicative form:

$$(\alpha\beta)_{ij} = \mu_{ij} - \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot j} + \bar{\mu}_{\cdot\cdot} = \lambda(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})(\bar{\mu}_{\cdot j} -$$

Then, you (indirectly) estimate λ and test whether $\lambda = 0$

Procedure:

① $P_j = \sum_{i=1}^{a} (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}) Y_{ij}$ ② $P = \sum_{j=1}^{b} P_j (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})$

③ $SS(\text{NONADDITIVITY}) = \dfrac{P^2}{\sum_i (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 \sum_j (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2}$

④ $SS(\text{Residual}) = SSE - SS(\text{NONADDITIVITY})$ →

ANOVA

Source	df	SS	MS
A	$a-1$	SSA	$MSA = \frac{SSA}{a-1}$
B	$b-1$	SSB	$MSB = \frac{SSB}{b-1}$
Error	$(a-1)(b-1)$	$SSE = \sum_i \sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$	$MSE = \frac{SSE}{(a-1)(b-1)}$
Nonadditivity	1	SS(NONADDITIVITY)	$MS(NA) = \frac{SS(NA)}{1}$
Residual	$(a-1)(b-1)-1$	$SSE - SS(NA) = SS(RES)$	$MS(RES) = \frac{SS(RES)}{(a-1)(b-1)-1}$

H_0: No Interaction (No nonadditivity)

H_A: Interaction (Nonadditivity)

T.S. $F_0 = \frac{MS(NA)}{MS(RES)}$

RR: $F_0 \geq F_{\alpha, 1, (a-1)(b-1)-1}$

$\cancel{P_1 = \sum_{i=1}^{a} \bar{y}_{i\cdot}}$

<u>Note</u>: Let $P_i = \sum_{j=1}^{4} (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) y_{ij}$

& $P = \sum_{i=1}^{3} P_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})$

$P_1 = 14(3.25) + 2(-1.75) + 1(-0.08) + 2(-1.42)$

$= 39.08$

$P_2 = 2(3.25) + 0(-1.75) + 2(-0.08) + 2(-1.42) = 3.50$

$P_3 = 2(3.25) + 1(-1.75) + 5(-0.08) + 0(-1.42) = 4.35$

$P = \sum_{i=1}^{3} P_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) = 39.08(2) + 3.50(-1.25) + 4.35(-0.75)$

$= 70.5225$

$\sum (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = (2.00)^2 + (-1.25)^2 + (-0.75)^2 = 6.125$

$\sum (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2 = (3.25)^2 + (-1.75)^2 + (-0.08)^2 + (-1.42)^2$

$= 15.6478$

$$SS(NA) = \frac{P^2}{\sum_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 \sum_j (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2} = \frac{(70.5225)^2}{6.125(15.6478)}$$

$= 51.89$

EXAMPLE - Based on Tukey's (1949) data

He begins w/ data

		B 1	2	3	4
	1	12	4	2	5
A	2	4	-2	-4	-5
	3	4	-3	-7	-2

This can be constructed from a model w/:

$\mu = 1 \quad \alpha_1 = 4 \quad \alpha_2 = \alpha_3 = -3 \quad \beta_1 = 6 \quad \beta_2 = 1 \quad \beta_3 = -4 \quad \beta_4 = 0$

& random perturbations for individual cells

He then squares data, divides by 10, and rounds to integers (This will cause interactions of the form: $\lambda (\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})(\bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot\cdot})$ presumably)

					$y_{i\cdot}$ Sums	$\bar{y}_{i\cdot}$ Means	$\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}$ Devs	ρ_i sum of cross produc.
	14	2	1	2	19	4.75	2.00	39.08
	2	0	2	2	6	1.50	-1.25	3.50
	2	1	5	0	8	2.00	-0.75	4.35
$y_{\cdot j}$ Sums	18	3	8	4	$33 = y_{\cdot\cdot}$		0	70.5225 = P
$\bar{y}_{\cdot j}$ Means	6.00	1.00	2.67	1.33		$2.75 = \bar{y}_{\cdot\cdot}$	6.125	
$\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}$ Deviations	3.25	-1.75	-0.08	-1.42	0	15.6478	51.89	

→

$$SS_{ERROR} = \sum_i \sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$$

i	j	y_{ij}	$\bar{y}_{i.}$	$\bar{y}_{.j}$	$\bar{y}_{..}$	$(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$
1	1	12	4.75	6.00	2.75	36
1	2	2	4.75	1.00	2.75	1
1	3	1	4.75	2.67	2.75	13.5
1	4	2	4.75	1.33	2.75	1.8
2	1	2	1.50	6.00	2.75	7.6
2	2	0	1.50	1.00	2.75	0.1
2	3	2	1.50	2.67	2.75	0.3
2	4	2	1.50	1.33	2.75	3.7
3	1	2	2.00	6.00	2.75	10.6
3	2	1	2.00	1.00	2.75	0.6
3	3	5	2.00	2.67	2.75	9.5
3	4	0	2.00	1.33	2.75	0.3

$SS_{ERROR} = 85.0$

$$SS(\text{RESIDUAL}) = SS_{ERROR} - SS(NA)$$

$$= 85.0 - 51.89 = 33.11$$

H_0: NO INTERACTION $\quad H_A$: INTERACTION (NON Additive)

$$\text{T.S. } F_0 = \frac{SS(NA)/1}{SS(RES)/[(a-1)(b-1)-1]} = \frac{51.89/1}{33.11/5} = 7.84$$

$$\text{RR: } F_0 \geq F_{.05,1,5} = 6.61$$

Method of Fitting Constants

Model w/ Interaction

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad i=1,\dots,a \quad j=1,\dots,b \quad k=1,\dots,r_{ij}$$

Model w/out interaction

$$Y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk} \qquad \left(\sum_i \hat{\alpha}_i = \sum_j \hat{\beta}_j = 0\right)$$

(A) Model w/OUT INTERACTION METHOD

(1) Fit 1-WAY ANOVA w/ $t = ab$ cells

$$\boxed{\begin{aligned} SS_{TRTS} &= \sum_i\sum_j r_{ij}\,\bar{Y}_{ij\cdot}^2 - \sum_i\sum_j r_{ij}\,\bar{Y}_{\cdots}^2 \\ SS_{ERROR} &= \sum_i\sum_j (r_{ij}-1)\,S_{ij}^2 \end{aligned}}$$

(2) Compute SSA, SSB (Ignoring remaining factor)

$$\boxed{\begin{aligned} SSA &= \sum_i r_{i\cdot}\,\bar{Y}_{i\cdot\cdot}^2 - \sum_i\sum_j r_{ij}\,\bar{Y}_{\cdots}^2 \\ SSB &= \sum_j r_{\cdot j}\,\bar{Y}_{\cdot j\cdot}^2 - \sum_i\sum_j r_{ij}\,\bar{Y}_{\cdots}^2 \end{aligned}}$$

(3) Obtain L.S. Estimates of μ, $\{\alpha_i\}$, $\{\beta_j\}$

$$Q = \sum_i\sum_j\sum_k \left(Y_{ijk} - \mu - \alpha_i - \beta_j\right)^2$$

$$\frac{\partial Q}{\partial \mu} = -2\sum_i\sum_j\sum_k \left(Y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j\right) = 0$$

$$\frac{\partial Q}{\partial \alpha_{i'}} = -2\sum_j\sum_k \left(Y_{i'jk} - \hat{\mu} - \hat{\alpha}_{i'} - \hat{\beta}_j\right) = 0$$

$$\frac{\partial Q}{\partial \beta_{j'}} = -2\sum_i\sum_k \left(Y_{ij'k} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_{j'}\right) = 0$$

(4) Obtain Reduction in Sum of squares due to "Fitting Constants"

$$\boxed{\hat{\mu}\,Y_{\cdots} + \sum_i \hat{\alpha}_i Y_{i\cdot\cdot} + \sum_j \hat{\beta}_j Y_{\cdot j\cdot}}$$

(5) Subtract off correction for Mean

$$\boxed{\hat{\mu}\,Y_{\cdots} + \sum_i \hat{\alpha}_i Y_{i\cdot\cdot} + \sum_j \hat{\beta}_j Y_{\cdot j\cdot} - \sum_i\sum_j r_{ij}\,\bar{Y}_{\cdots}^2}$$

(6) Obtain SS for each factor adjusted for other factor

- A adjusted for B: Result from (5) $- SSB = SS(A|B)$
- B adjusted for A: " " " $- SSA = SS(B|A)$

(7) Set up ANOVA Table (w/out Interaction)

Source	df	SS	MS	F
Factors A & B	$(a-1)+(b-1)$	Result (5)	—	—
A | B	$a-1$	$SS(A\|B)$	$MS(A\|B) = \frac{SS(A\|B)}{a-1}$	$\frac{MS(A}{MSE}$
B | A	$b-1$	$SS(B\|A)$	$MS(B\|A) = \frac{SS(B\|A)}{b-1}$	$\frac{MS(B\|}{MSE}$
ERROR	$\sum_i\sum_j (r_{ij}-1)$	SSE	$MSE = \frac{SSE}{\sum_i\sum_j (r_{ij}-1)}$	—
TOTAL				

s
or
additive

(FC 3)

		Ad: Read ($j=1$)	Surv ($j=2$)	$Y_{i..}$	$\bar{Y}_{i..}$
Program Involvement	LOW ($i=1$)	$\bar{Y}=3.40$, $s=1.25$, $r=7$	$\bar{Y}=3.70$, $s=1.27$, $r=9$	57.10	3.56875
	HIGH ($i=2$)	$\bar{Y}=4.48$, $s=1.36$, $r=8$	$\bar{Y}=2.48$, $s=0.47$, $r=7$	53.20	3.54667
	$Y_{.j.}$	59.64	50.66	110.30	3.55806
	$\bar{Y}_{.j.}$	3.97600	3.16625		

① $SSTRTS = 7(3.40)^2 + 9(3.70)^2 + 8(4.48)^2 + 7(2.48)^2 - 31(3.55806)^2$

$= 80.9200 + 123.2100 + 160.5632 + 43.0528 - 392.4535$

$= 407.7460 - 392.4535 = 15.2925$

$SSE = 6(1.25)^2 + 8(1.27)^2 + 7(1.36)^2 + 6(0.47)^2 = 36.55080$

② $SSA = 16\left[(3.56875 - 3.55806)^2\right] + 15\left((3.54667 - 3.55806)^2\right]$

$= .001828 + .001946 = .003774$

$SSB = 15(3.976 - 3.55806)^2 + 16(3.16625 - 3.55806)^2$

$= 2.620108 + 2.456241 = 5.076349$

③ NORMAL EQUATIONS

$\mu: \; 31\hat{\mu} + 16\hat{\alpha}_1 + 15\hat{\alpha}_2 + 15\hat{\beta}_1 + 16\hat{\beta}_2 = 110.30$

$\alpha_1: \; 16\hat{\mu} + 16\hat{\alpha}_1 + 7\hat{\beta}_1 + 9\hat{\beta}_2 = 57.10$

$\alpha_2: \; 15\hat{\mu} + 15\hat{\alpha}_2 + 8\hat{\beta}_1 + 7\hat{\beta}_2 = 53.20$

$\beta_1: \; 15\hat{\mu} + 7\hat{\alpha}_1 + 8\hat{\alpha}_2 + 15\hat{\beta}_1 = 59.64$

$\beta_2: \; 16\hat{\mu} + 9\hat{\alpha}_1 + 7\hat{\alpha}_2 + \; + 16\hat{\beta}_2 = 50.66$

SOLVING FOR α's in terms of μ's & β's

$16\hat{\alpha}_1 = 57.10 - 16\hat{\mu} - 7\hat{\beta}_1 - 9\hat{\beta}_2$

$15\hat{\alpha}_2 = 53.20 - 15\hat{\mu} - 8\hat{\beta}_1 - 7\hat{\beta}_2$

Putting α's into β equations

$$15\hat{\mu} + 7\left(\frac{57.10}{16} - \hat{\mu} - \frac{7}{16}\hat{\beta}_1 - \frac{9}{16}\hat{\beta}_2\right)$$

$$+ 8\left(\frac{53.20}{15} - \hat{\mu} - \frac{8}{15}\hat{\beta}_1 - \frac{7}{15}\hat{\beta}_2\right) + 15\hat{\beta}_1 = 59.64$$

$$\Rightarrow \left[\left(\frac{-49}{16}\right) + \left(\frac{-64}{15}\right) + 15\right]\hat{\beta}_1 + \left[\left(-\frac{63}{16}\right) + \left(\frac{-56}{15}\right)\right]\hat{\beta}_2 = 6.285417$$

$$\Rightarrow 7.670833\hat{\beta}_1 - 7.670833\hat{\beta}_2 = 6.285417$$

Setting $\hat{\beta}_2 = -\hat{\beta}_1 \quad (\Sigma\hat{\beta}_j = 0) \Rightarrow 15.341667\hat{\beta}_1 = 6.285417$

$$\Rightarrow \hat{\beta}_1 = 0.409696 \quad \Rightarrow \hat{\beta}_2 = -0.409696$$

Putting β's back into α Equation

$$16\hat{\mu} + 16\hat{\alpha}_1 = 57.10 - 7(.409696) - 9(-.409696) = 57.919392$$

$$\Rightarrow \hat{\mu} + \hat{\alpha}_1 = 3.619962$$

$$15\hat{\mu} + 15\hat{\alpha}_2 = 53.20 - 8(.409696) - 7(-.409696) = 52.790304$$

$$\Rightarrow \hat{\mu} + \hat{\alpha}_2 = 3.519354$$

Setting $\hat{\alpha}_2 = -\hat{\alpha}_1 \quad (\Sigma\hat{\alpha}_i = 0)$

$$\Rightarrow \hat{\mu} + \hat{\alpha}_1 = 3.619962$$
$$\hat{\mu} - \hat{\alpha}_1 = 3.519354$$

$$\Rightarrow 2\hat{\alpha}_1 = 0.100608 \Rightarrow \hat{\alpha}_1 = .050304 \Rightarrow \hat{\alpha}_2 = -.050304$$

$$\Rightarrow \hat{\mu} = 3.569658$$

(4) & (5) Reduction in SS due to Fitting constants & subtracting off correction for mean

$$\hat{\mu}Y_{\cdots} + \sum_i \hat{\alpha}_i Y_{i\cdot\cdot} + \sum_j \hat{\beta}_j Y_{\cdot j\cdot} - N\bar{Y}_{\cdots}^2$$

$$= 3.569658(110.30) + (.050304)(57.10) + (-.050304)(53.20) + (.409696)(59.64)$$
$$+ (-.409696)(50.66) = \cancel{3\ldots.608533} = 5.155013$$

6 $SS(A|B) = 5.155013 - 5.076349 = .078664$

$SS(B|A) = 5.15503 - .003774 = 5.151256$

7

Source	df	SS	MS	F	$F_{.05,1,27}$
A : B	1+1=2	5.155013	—	—	
A | B	1	.078664	.078664	.0581	4.21
B | A	1	5.151256	5.151256	3.8052	4.21
ERROR	31-4=27	36.55080	1.353733	—	

TESTING FOR INTERACTIONS

8 Obtain $SS(AB|A,B) =$ SS_{TRTS} – Result from 5

$\Rightarrow SS(AB|A,B) = 15.2925 - 5.155013 = 10.1375$

$$\Rightarrow F_{AB} = \frac{MS(AB|A,B)}{MSE} = \frac{10.1375/1}{1.353733} = 7.4885 > 4.21$$