Chapter 5 - Random Effects Model

- Groups of Experimental Units are sampled from a population of such groups

- Model: $Y_{ij} = \mu + a_i + e_{ij}$ $\quad i = 1, \dots, t \quad j = 1, \dots, r$

$\boxed{\text{Variance Components Model}}$

$e_{ij} \sim NID(0, \sigma_e^2) \qquad a_i \sim NID(0, \sigma_a^2) \qquad e_{ij} \perp a_i$

$0 = \sigma_a^2 \Rightarrow$ All group effects are equal

$0 < \sigma_a^2 \Rightarrow$ Group effects vary in population

$$E[Y_{ij}] = E[\mu + a_i + e_{ij}] = E[\mu] + E[a_i] + E[e_{ij}] = \mu$$

$$\sigma_Y^2 = V(Y_{ij}) = V(\mu + a_i + e_{ij}) = V(a_i) + V(e_{ij}) = \sigma_a^2 + \sigma_e^2$$

$\underline{i = i'}$

$$Cov(Y_{ij}, Y_{ij'}) = Cov(\mu + a_i + e_{ij}, \mu + a_i + e_{ij'})$$

$$= Cov(a_i + e_{ij}, a_i + e_{ij'}) = Cov(a_i, a_i) + Cov(a_i, e_{ij'})$$

$$+ Cov(e_{ij}, a_i) + Cov(e_{ij}, e_{ij'}) = V(a_i) + 0 + 0 + 0 = \sigma_a^2$$

$i \neq i'$, $\underline{j = j' \text{ or } j \neq j'}$

$$Cov(Y_{ij}, Y_{i'j'}) = Cov(\mu + a_i + e_{ij}, \mu + a_{i'} + e_{i'j'})$$

$$= Cov(a_i + e_{ij}, a_{i'} + e_{i'j'}) = Cov(a_i, a_{i'}) + Cov(a_i, e_{i'j'})$$

$$+ Cov(e_{ij}, a_{i'}) + Cov(e_{ij}, e_{i'j'}) = 0 + 0 + 0 + 0$$

$$\Rightarrow \boxed{Cov(Y_{ij}, Y_{i'j'}) = \begin{cases} \sigma_a^2 + \sigma_e^2 & i = i', j = j' \\ \sigma_a^2 & i = i' \ \ j \neq j' \\ 0 & i \neq i' \end{cases}}$$

Expected Mean Squares

Among Groups $\quad SSA = r\sum_{i=1}^{t}(\bar{y}_{i.}-\bar{y}_{..})^2 = r\sum_{i=1}^{t}\bar{y}_{i.}^2 - rt\,\bar{y}_{..}^2$

Within Groups $\quad SSW = \sum_{i=1}^{t}\sum_{j=1}^{r}(y_{ij}-\bar{y}_{i.})^2 = \sum_i\sum_j y_{ij}^2 - r\sum_{i=1}^{t}\bar{y}_{i.}^2$

$$\boxed{E[y_{ij}] = \mu, \quad V[y_{ij}] = \sigma_a^2+\sigma_e^2 \Rightarrow E[y_{ij}^2] = \mu^2+\sigma_a^2+\sigma_e^2}$$

$$\bar{y}_{i.} = \frac{1}{r}\sum_{j=1}^{r} y_{ij}$$

$$E[\bar{y}_{i.}] = \frac{1}{r}\sum_{j=1}^{r}E[y_{ij}] = \frac{1}{r}\sum_{j=1}^{r}\mu = \frac{1}{r}(r\mu) = \mu$$

$$V[\bar{y}_{i.}] = V\left[\frac{1}{r}\sum_{j=1}^{r}y_{ij}\right] = \frac{1}{r^2}V\left[\sum_{j=1}^{r}y_{ij}\right]$$

$$= \frac{1}{r^2}\left[\sum_{j=1}^{r}V(y_{ij}) + 2\sum\sum_{j<j'}\mathrm{cov}(y_{ij}, y_{ij'})\right]$$

$$= \frac{1}{r^2}\left[\sum_{j=1}^{r}(\sigma_a^2+\sigma_e^2) + 2\binom{r}{2}\sigma_a^2\right] = \frac{1}{r^2}\left[r(\sigma_a^2+\sigma_e^2)+r(r-1)\sigma_a^2\right]$$

$$\Rightarrow \boxed{E[\bar{y}_{i.}^2] = \mu^2 + \frac{1}{r^2}\left[r(\sigma_a^2+\sigma_e^2)+r(r-1)\sigma_a^2\right]}$$

$$\bar{y}_{..} = \frac{1}{rt} \sum_{i=1}^{t}\sum_{j=1}^{r} y_{ij}$$

$$E[\bar{y}_{..}] = \frac{1}{rt} \sum_i\sum_j E[y_{ij}] = \frac{1}{rt} rt\mu = \mu$$

$$V[\bar{y}_{..}] = \left(\frac{1}{rt}\right)^2 V\left[\sum_i\sum_j y_{ij}\right]$$

$$= \left(\frac{1}{rt}\right)^2 \Big\{ \overset{①}{\sum_i\sum_j V(y_{ij})} + \overset{②}{2\sum_i \sum_{j<j'} Cov(y_{ij}, y_{ij'})}$$

$$+ \underset{③}{2 \sum_{i<i'}\sum_j\sum_{j'} Cov(y_{ij}, y_{i'j'})} \Big\}$$

NOTE: Term 1 is made up of tr terms each $= \sigma_e^2 + \sigma_a^2$

Term 2 is made up of $t\binom{r}{2}$ terms each $= \sigma_a^2$

Term 3 is made up of $r^2\binom{t}{2}$ terms each $= 0$

$$= \left(\frac{1}{rt}\right)^2 \left[tr(\sigma_e^2 + \sigma_a^2) + 2t \frac{r(r-1)}{2} \sigma_a^2 + 2r^2\binom{t}{2} \cdot 0 \right]$$

$$= \left[\frac{1}{rt}\right]^2 \left\{ rt(\sigma_e^2 + \sigma_a^2) + tr(r-1)\sigma_a^2 \right\}$$

$$\Rightarrow \boxed{E[\bar{y}_{..}^2] = \mu^2 + \left(\frac{1}{rt}\right)^2 \left\{ rt(\sigma_e^2 + \sigma_a^2) + tr(r-1)\sigma_a^2 \right\}}$$

$$E[SSA] = E\left\{ r\sum_i \bar{y}_{i\cdot}^2 - rt\,\bar{y}_{\cdot\cdot}^2 \right\}$$

$$= r\sum_i E[\bar{y}_{i\cdot}^2] - rt\,E[\bar{y}_{\cdot\cdot}^2]$$

$$= r\sum_{i=1}^{t}\left\{ \mu^2 + \frac{1}{r^2}\left[r(\sigma_a^2+\sigma_e^2) + r(r-1)\sigma_a^2 \right]\right\}$$

$$-rt\left\{ \mu^2 + \left(\frac{1}{rt}\right)^2 \left\{ rt(\sigma_e^2+\sigma_a^2) + tr(r-1)\sigma_a^2 \right\}\right.$$

$$= \left[rt\mu^2 + \left(\frac{r^2}{r^2}\right)t(\sigma_a^2+\sigma_e^2) + \frac{r^2(r-1)}{r^2}\,t\sigma_a^2 \right]$$

$$- \left[rt\mu^2 + \frac{(rt)^2}{(rt)^2}(\sigma_e^2+\sigma_a^2) + \frac{(rt)^2}{(rt)^2}(r-1)\sigma_a^2 \right]$$

$$= \mu^2[rt-rt] + (\sigma_a^2+\sigma_e^2)(t-1) + \sigma_a^2[t(r-1)-(r-1)]$$

$$= \sigma_a^2(t-1) + \sigma_e^2(t-1) + \sigma_a^2(t-1)(r-1)$$

$$= \boxed{\sigma_a^2\, r(t-1) + \sigma_e^2(t-1) = E[SSA]}$$

$$\Rightarrow \boxed{E[MSA] = E\left[\frac{SSA}{t-1}\right] = r\sigma_a^2 + \sigma_e^2}$$

$$E[SS] = E\left\{ \sum_i \sum_j y_{ij}^2 - r \sum_i \bar{y}_{i\cdot}^2 \right\}$$

$$= \underbrace{\sum_i \sum_j E[y_{ij}^2]}_{1} - \underbrace{r \sum_i E[\bar{y}_{i\cdot}^2]}_{2}$$

$$2 = rt\mu^2 + t(\sigma_a^2 + \sigma_e^2) + t(r-1)\sigma_a^2$$

(see previous page)

$$1 = rt\left[\mu^2 + \sigma_a^2 + \sigma_e^2\right]$$

$$1 - 2 = \mu^2[rt - rt] + \sigma_a^2\left[rt - [t + t(r-1)]\right] + \sigma_e^2[rt - t]$$

$$= \mu^2(0) + \sigma_a^2 \overbrace{\left[rt - t[1 + r - 1]\right]}^{0} + \sigma_e^2 t(r-1)$$

$$\Rightarrow \boxed{E[SSW] = \sigma_e^2 t(r-1)}$$

$$\Rightarrow \boxed{E[MSW] = E\left[\frac{SSW}{t(r-1)}\right] = \sigma_e^2}$$

Unbiased (ANOVA) ESTIMATES OF VARIANCE COMPONEN[TS]

$$\hat{\sigma}_e^2 = MSW$$

$$\hat{\sigma}_a^2 = \frac{MSA - MSW}{r}$$

$$\hat{\sigma}_y^2 = \hat{\sigma}_a^2 + \hat{\sigma}_e^2 = MSW + \frac{MSA - MSW}{r}$$

$$= \frac{(r-1)\,MSW + MSA}{r}$$

$H_0: \sigma_a^2 = 0 \qquad H_A: \sigma_a^2 > 0$

T.S. $F_0 = \dfrac{MSA}{MSW}$

RR: $F_0 \geq F_{\alpha, t-1, N-t}$

P-VALUE $= Pr\{F \geq F_0 \mid F \sim F_{t-1, N-t}\}$

$(1-\alpha)100\%$ CI for σ_e^2

$$\frac{SSW}{\sigma_e^2} \sim \chi^2_{N-t} \Rightarrow Pr\left\{ \chi^2_{1-\frac{\alpha}{2}, N-t} \le \frac{SSW}{\sigma_e^2} \le \chi^2_{\frac{\alpha}{2}, N-t} \right\} = 1-\alpha$$

$$\Rightarrow Pr\left\{ \frac{1}{\chi^2_{1-\frac{\alpha}{2}, N-t}} \ge \frac{\sigma_e^2}{SSW} \ge \frac{1}{\chi^2_{\frac{\alpha}{2}, N-t}} \right\} = 1-\alpha$$

$$\Rightarrow Pr\left\{ \frac{SSW}{\chi^2_{1-\frac{\alpha}{2}, N-t}} \ge \sigma_e^2 \ge \frac{SSW}{\chi^2_{\frac{\alpha}{2}, N-t}} \right\} = 1-\alpha$$

$(1-2\alpha)100\%$ CI for σ_a^2

$$\frac{SSA\left(1 - F_{\alpha/2, t-1, N-t} / F_0\right)}{r\,\chi^2_{\frac{\alpha}{2}, t-1}} < \sigma_a^2 < \frac{SSA\left(1 - F_{1-\frac{\alpha}{2}, t-1, N-t} / F_0\right)}{r\,\chi^2_{1-\frac{\alpha}{2}, t-1}}$$

where $F_{1-\frac{\alpha}{2}, t-1, N-t} = \dfrac{1}{F_{\frac{\alpha}{2}, N-t, t-1}}$

Negative Variance Component ESTIMATES

While true variance components are nonnegative, estimates can be negative when MSA < MSE

Some possibilities

1. Take negative estimate ~~that~~ to indicate that true value is 0.
2. Retain it (strange subsequent calculations)
3. Sign that model is misspecified
4. Use other method of estimation (Many exist)
5. Collect more data.

Intraclass Correlation

$$\rho_I = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2} = \frac{\sigma_a^2}{\sigma_y^2}$$

$$\hat{\rho}_I = \frac{\hat{\sigma}_a^2}{\hat{\sigma}_y^2} = \frac{MSA - MSW}{MSA + (r-1)MSW}$$

$(1-\alpha)100\%$ CI for ρ_I

$$\left[\frac{F_0 - F_{\frac{\alpha}{2}, t-1, N-t}}{F_0 + (r-1) F_{\frac{\alpha}{2}, t-1, N-t}} , \frac{F_0 - F_{(1-\frac{\alpha}{2}), t-1, N-t}}{F_0 + (r-1) F_{(1-\frac{\alpha}{2}, t-1, N-t)}} \right]$$

Sample Size Calculations

Goal: Determine r so that

$$Pr\{\text{Conclude } H_A : \sigma_a^2 > 0 \mid \sigma_a^2 = c > 0\} = 1-\beta$$

When H_A is true: $F = \frac{MSA}{MSW} \lambda^2 \sim F_{\nu_1, \nu_2}$

where $\lambda^2 = 1 + \frac{r\sigma_a^2}{\sigma_e^2}$

$$\Rightarrow 1-\beta = Pr\left\{F > \frac{1}{\lambda^2} F_{\alpha, \nu_1, \nu_2}\right\}$$

Algorithm

0. Select σ_a^2, $(1-\beta)$, $\alpha = .05$
1. Choose $r = 2$
2. $\nu_1 = t-1$, $\nu_2 = t(r-1)$
3. Compute $\lambda^2 = 1 + \frac{r\sigma_a^2}{\hat{\sigma}_e^2}$ (or specify $\frac{\sigma_a^2}{\sigma_e^2}$) $\longrightarrow$ Some prior estimate
4. Compute $\frac{1}{\lambda^2} F(.95, \nu_1, \nu_2)$ $\boxed{\text{FINV Function}}$
5. Compute $Pr\{F_{\nu_1, \nu_2} \geq (4)\}$
6. IF (5) $\geq 1-\beta$ stop otherwise go to (2) & increment r.