Factorial Designs

Part A. Fixed Effects

QA.1. A hotel is interested in studying the effects of washing machines and detergents on whiteness of bed sheets. The hotel has 4 washing machines and 3 brands of detergent. They randomly assign n=4 sheets for each combination of machine and detergent (each sheet is only observed for one combination of machine and detergent). After washing, the sheets are measured for whiteness (high scores are better). The model fit is:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad e_{ijk} \sim NID(0, \sigma^2) \qquad \sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} (\alpha\beta)_{ij} = \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0$$

Washer\Detergent	1	2	3	Mean
1	25	30	35	30
2	20	35	65	40
3	25	45	50	40
4	30	50	70	50
Mean	25	40	55	40

pA.1.a. Give least squares estimates of the following parameters:

α_1: β_1:

$(\alpha\beta)_{11}$: σ^2:

Source	df	SS	MS	F	F(0.05)
Machine					
Detergent					
M*D					
Error		36000		#N/A	#N/A
Total		57000	#N/A	#N/A	#N/A

pA.1.b. Complete the ANOVA table.

pA.1.c. Is there a significant interaction between Machine and Detergent on whiteness scores? **Yes / No**

pA.1.d. Is there a significant main effect for Machines? **Yes / No**

pA.1.e. Is there a significant main effect for Detergents? **Yes / No**

QA.2. For the 2-way ANOVA with fixed effects and interaction, considering the following two models:

1) $y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,a;\ j = 1,\ldots,b;\ k = 1,\ldots,r \qquad \sum_i \alpha_i = \sum_j \beta_j = \sum_i \alpha\beta_{ij} = \sum_j \alpha\beta_{ij} = 0$

2) $y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$

pA.2.a. Write out the Sums of Squares and Degrees of Freedom for each model:

Model 1: SSA = df_A =

SSB = df_B

SSAB = df_{AB} =

Model 2: SSTrts = df_{Trts} =

pA.2.b. (Algebraically) Show that SSA + SSB + SSAB = SSTrts and that $df_A + df_B + df_{AB} = df_{Trts}$

Hint: The expansion for SSAB is very much like the expansion for SSA.

QA.3. A 2-Factor (fixed effects ANOVA) is fit, with factor A at 5 levels, and factor B at 3 levels. There were 5 replicates for each combination of factor levels. Compute Bonferroni's and Tukey's minimum significant differences for comparing all pairs of means, each (Factors A and B, respectively) with an experiment-wise error rate of 0.05. Note: This is the same as the margin of error for the point estimates (half-width of simultaneous CI's). Assume that interaction is not present, and give your results as functions of only MSE.

Factor:	A	B
Bonferroni		
Tukey		

QA.4. An unbalanced 2-Way ANOVA was fit (based on n_T = 433 people), relating music empathizing scores (Y) to the factors: **E**xperience (Professional, Amateur, Non-Musician) and **G**ender (Male, Female). Due to unbalanced sample sizes for the 6 cells (sample sizes ranged from 38 to 160), a regression model was fit with the following dummy coded variables:

$$X_{\text{Pro}} = \begin{cases} 1 \text{ if Professional} \\ -1 \text{ if Non-Musician} \\ 0 \text{ if Amateur} \end{cases} \qquad X_{\text{Amateur}} = \begin{cases} 1 \text{ if Amateur} \\ -1 \text{ if Non-Musician} \\ 0 \text{ if Professional} \end{cases} \qquad X_{\text{Female}} = \begin{cases} 1 \text{ if Female} \\ \text{-1 if Male} \end{cases} \qquad X_{P*F} = X_{\text{Pro}} X_{\text{Female}} \qquad X_{A*F} = X_{\text{Amateur}} X_{\text{Female}}$$

The following output gives partial ANOVA Results for 4 models: (1:E,G,EG 2:E,G 3:G,EG 4:E,EG)

Model	Predictor Variables	SSE
1	X_P, X_A, X_F, X_PF, X_AF	52949
2	X_P, X_A, X_F	53433
3	X_F, X_PF, X_AF	53454
4	X_P, X_A, X_PF, X_AF	54397

Use these models to test the following hypotheses (always round down on error degrees of freedom, if necessary):

pA.4.a. H_0: No Experience by Gender Interaction

pA.4.a.i. Test Statistic: ______________________ p.5.a.ii. Reject H_0 if Test Stat is in the range ________________

pA.4.b. H_0: No Experience Main Effect (Controlling for Gender and EG)

pA.4.b.i. Test Statistic: ______________________ p.5.b.ii. Reject H_0 if Test Stat is in the range ________________

pA.4.c. H_0: No Gender Main Effect (Controlling for Experience and EG)

pA.4.c.i. Test Statistic: ______________________ p.5.c.ii. Reject H_0 if Test Stat is in the range ________________

QA.5. An experiment was conducted to determine the effects of food appearance on the plate (balanced/unbalanced and color/monochrome) on consumers hedonic liking of food. A sample of 68 restaurant customers was obtained, and assigned such that 17 received each combination of balance and color. The following table gives the means (SDs) for each treatment. The response (Y) was hedonic liking of the food dish (on a scale of -100 to +100).

Model: $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \quad i=1,2 \;\; j=1,2 \;\; k=1,...,17 \quad \sum_{i=1}^{2}\alpha_i = \sum_{j=1}^{2}\beta_j = \sum_{i=1}^{2}(\alpha\beta)_{ij} = \sum_{j=1}^{2}(\alpha\beta)_{ij} = 0 \quad \varepsilon_{ijk} \sim N(0,\sigma^2)$

pA.5.a. The following table gives the means (SDs) for each treatment:

Balance\Color	1=Mono	2=Color	Row Mean
1=Balanced	13.3 (53)	-1.7 (69)	
2=Unbalanced	13.6 (41)	2.2 (58)	
Col Mean			

Complete the following ANOVA table:

ANOVA					
Source	**df**	**SS**	**MS**	**F***	**F(0.95)**
Balance					
Color					
Bal*Col					
Error				**#N/A**	**#N/A**
Total			**#N/A**	**#N/A**	**#N/A**

pA.5.b. Test H_0: No Interaction between Balance and Color

pA.5.b.i. Test Stat: _______ pA.5.b.ii. Reject H_0 if Test Stat is in the range _________ pA.5.b.iii. P-value **> or <** .05?

pA.5.c. Test H_0: No Balance effect

pA.5.c.i. Test Stat: _______ pA.5.c.ii. Reject H_0 if Test Stat is in the range _________ pA.5.c.iii. P-value **> or <** .05?

pA.5.d. Test H_0: No Color effect

pA.5.d.i. Test Stat: _________ pA.5.d.ii. Reject H_0 if Test Stat is in the range ________ pA.5.d.iii. P-value **> or <** .05?

QA.6. An experiment was conducted to compare *a* = 3 theories for the apparent modulus of elasticity (*Y*) of *b* = 3 apple varieties. The 3 theories were: Hooke's, Hertz's, and Boussineq's; the 3 apple varieties were: Golden Delicious, Red Delicious, and Granny Smith. The researchers determined the elasticity for *n* = 15 based on each combination of theory and variety. For the purposes of this experiment, each factor is fixed.

$$\text{Model: } Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad \varepsilon_{ijk} \sim NID(0,\sigma^2) \quad \sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = \sum_{i=1}^{a}(\alpha\beta)_{ij} = \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{ij\bullet}\right)^2 = 17.095 \qquad \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{\bullet\bullet\bullet}\right)^2 = 113.119$$

Cell Means	GoldenDelicious	RedDelicious	GrannySmith	Row Mean
Hooke	2.68	3.46	4.23	3.457
Hertz	2.44	3.06	3.84	3.113
Boussinesq	1.53	1.89	2.36	1.927
Column Mean	2.217	2.803	3.477	2.832

Complete the following Analysis of Variance Table, and test for interaction effects and main effects.

Source	df	SS	MS	F	F(.95)	P-value
Theory						>0.05 or <0.05
Variety						>0.05 or <0.05
Theory*Variety						>0.05 or <0.05
Error				#N/A	#N/A	#N/A
Total			#N/A	#N/A	#N/A	#N/A

QA.7 Consider a 2-factor, fixed effects, interaction model with *a* = *b* = *n* = 2.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad \varepsilon_{ijk} \sim NID(0,\sigma^2) \quad \sum_{i=1}^{2}\alpha_i = \sum_{j=1}^{2}\beta_j = \sum_{i=1}^{2}(\alpha\beta)_{ij} = \sum_{j=1}^{2}(\alpha\beta)_{ij} = 0$$

pA.7.a. Given the following data, obtain MS_{Err}.

Y111	Y112	Y121	Y122	Y211	Y212	Y221	Y222
22	18	31	29	8	12	37	43

SS_{Err} = ______________________________ MS_{Err} = _________________________

pA.7.b. Use the matrix form of the model to obtain: the least squares estimate of the parameter vector $\boldsymbol{\beta}^*$, its estimated variance-covariance matrix, standard errors, and t-tests for all of the parameters.

$$\boldsymbol{\beta}^* = \begin{bmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ (\alpha\beta)_{11} \end{bmatrix} \qquad \mathbf{X} = \qquad \mathbf{X'X} =$$

$$(\mathbf{X'X})^{-1} = \qquad \mathbf{X'Y} =$$

$$\hat{\boldsymbol{\beta}}^* = \qquad \hat{V}\left\{\hat{\boldsymbol{\beta}}^*\right\} =$$

Parameter	Estimate	Std Error	t	t(.975)
μ				
$\alpha 1$				
$\beta 1$				
$(\alpha\beta)11$				

QA.7. A 2-Way fixed effects model is fit, with factor A at 4 levels, and factor B at 3 levels. There are 3 replicates for each combinations of factors A and B. The error sum of squares is SS_{Err} = 720.

pA.7.a. Suppose that the interaction is not significant. What will be Tukey's and Bonferroni's minimum significant differences be for comparing the means for factor A (averaged across levels of factor B).

Tukey's HSD: ______________________________ Bonferroni's MSD: ___________________________

pA.7.b. Suppose that the interaction is significant. What will be Tukey's and Bonferroni's minimum significant differences be for comparing the means for factor A (within a particular level of factor B).

Tukey's HSD: ______________________________ Bonferroni's MSD: ___________________________

pA.7.c. How large would the interaction sum of squares, SS_{AB} need to be to reject H_0: No interaction between factors A&B?

QA.8. A forensic researcher is interested in the ridge densities between males and females and Caucasians and African-Americans. Samples of r=100 from each sub-population are obtained and for each subject, the number of ridges per 25mm^2 is measured. Means (standard deviations) are given in the following table.

Gender\Race	Caucasian	African-American
Female	**13.40 (1.24)**	**12.60 (1.43)**
Male	**11.10 (1.31)**	**10.90 (1.15)**

Complete the following ANOVA table, providing tests for interaction, as well as main effects for gender and race.

Source	df	SS	MS	F	F(0.05)	Significant Effects?
Gender						
Race						
Gender*Race						
Error						
Total						

QA.9. Researchers measured water accumulation in multi-layer clothing. They considered 3 types of outer layers (wool, polyester, down) and the same 3 types of inner layers (wool, polyester, down). They had 2 replicates at each combination of outer and inner layers.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad e_{ijk} \sim NID(0, \sigma^2)$$

pA.9.a. Complete the following ANOVA table.

ANOVA					
Source	**df**	**SS**	**MS**	**F**	**F(0.05)**
Outer		**13.62**			
Inner		**75.37**			
Interaction		**35.04**			
Error					
Total		**134.26**			

pA.9.b. The P-value for testing H_0: $(\alpha\beta)_{11} = \ldots = (\alpha\beta)_{33} = 0$ is **> 0.05** **< 0.05** **(Circle one)**

pA.9.c. The following table gives the cell means for all combinations of outer (rows) and inner (columns). Use Tukey's method to compare all pairs of treatments (combinations of outer and inner layers), and draw lines connecting all treatments that are not significantly different. Note: W/P $\Rightarrow$ Outer = Wool, Inner = Polyester

Outer\Inner	Wool	Polyester	Down
Wool	22.35	25.40	25.40
Polyester	21.55	23.65	26.20
Down	20.50	19.15	27.30

Tukey's HSD = ____

D/P D/W P/W W/W P/P W/P W/D P/D D/D

QA.10. An experiment was conducted to determine the effects of viewing a magical film versus a non-magical film in children. Samples of 32 6-year olds, and 32 8-year-olds were selected, and randomly assigned such that 16 of each age-group viewed the magical film and 16 of each age-group viewed the non-magical. The following table gives the means (SDs) for each treatment. The response (Y) was a score on an imagination scale (rating of a child acting out an object or animal).

Model: $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \quad i=1,2 \; j=1,2 \; k=1,...,16 \quad \sum_{i=1}^{2}\alpha_i = \sum_{j=1}^{2}\beta_j = \sum_{i=1}^{2}(\alpha\beta)_{ij} = \sum_{j=1}^{2}(\alpha\beta)_{ij} = 0 \quad e_{ijk} \sim N(0,\sigma^2)$

The following table gives the means (SDs) for each treatment:

Age\Film	Non-Magical	Magical	Mean
6	17.0 (2.7)	21.6 (4.1)	19.3
8	18.7 (3.8)	22.7 (3.8)	20.7
Mean	17.85	22.15	20

Complete the following ANOVA table:

Source	df	SS	MS	F	F(.05)
Age					
Film					
A*F					
Error				#N/A	#N/A
Total			#N/A	#N/A	#N/A

QA.11. An unbalanced two-way ANOVA was conducted to compare ethics scores on an exam (Y) among members of the U.S. Coast Guard. The factors were Gender (X_1=1 if Male, -1 if Female) and Rank (X_2=1 if Officers, -1 if Enlisted). The sample sizes were: M/O = 72, M/E = 180, F/O = 15, F/E =32. Four regressions models were fit:

Model 1: $E\{Y_{ijk}\} = \beta_0 + \beta_1 X_{1ijk} + \beta_2 X_{2ijk} + \beta_3 X_{1ijk} X_{2ijk}$ $\quad SSE_1 = 42088 \quad \hat{Y}_1 = 36.35 - 2.40X_1 + 3.85X_2 - 0.50X_1X_2$

Model 2: $E\{Y_{ijk}\} = \beta_0 + \beta_1 X_{1ijk} + \beta_2 X_{2ijk}$ $\quad SSE_2 = 42122 \quad \hat{Y}_2 = 36.23 - 2.21X_1 + 3.52X_2$

Model 3: $E\{Y_{ijk}\} = \beta_0 + \beta_2 X_{2ijk} + \beta_3 X_{1ijk} X_{2ijk}$ $\quad SSE_3 = 42873 \quad \hat{Y}_3 = 34.75 + 3.30X_2 + 0.39X_1X_2$

pA.11.a. Test whether there is an interaction between Gender and Rank. H_0: ____________ H_A: ____________

Test Statistic ________________________ Rejection Region ________________________ P-value **> 0.05** or **< 0.05** pA.11.c. Test whether there is main effect for Gender. H_0: ____________ H_A: ___________

Test Statistic _______________________ Rejection Region _______________________ P-value **> 0.05** or **< 0.05**

pA.11.c. Based on Model 2, give the predicted scores for all combinations of Gender and Rank.

Gender\Rank	Officer	Enlisted
Male		
Female		

QA.12. A study was conducted, measuring the effects of 3 electronic **Readers** and 4 **Illumination** levels on time for people to read a given text (100s of seconds). There were a total of 60 subjects, 5 assigned to each combination of Reader/Illumination level. For this analysis, consider both Reader and Illumination level as fixed effects.

pA.12.a Complete the following ANOVA table, and test for significant Reader/Illumination Interaction effects, as well as main effects for Reader and Illumination levels.

ANOVA						
Source	df	SS	MS	F	F(0.05)	Significant Effects?
Reader		70.70				
Illumination		148.11				
Read*Illum		2.15				
Error		365.02				
Total		585.98				

pA.12.b Use Tukey's Method to make all pairwise comparisons among Readers.

$\overline{Y}_{1\bullet\bullet} = 12.53 \quad \overline{Y}_{2\bullet\bullet} = 10.32 \quad \overline{Y}_{3\bullet\bullet} = 10.14$

pA.12.c Use Bonferroni's method to make all pairwise comparisons among Illumination levels.

$\overline{Y}_{\bullet 1\bullet} = 13.07 \quad \overline{Y}_{\bullet 2\bullet} = 11.95 \quad \overline{Y}_{\bullet 3\bullet} = 9.71 \quad \overline{Y}_{\bullet 4\bullet} = 9.26$

QA.13. A study compared the effects of a = 3 brands of squash rackets (Factor A: med, high, and higher price) and b = 2 levels of strings (Factor B: factory or new) the speed of a ball's bounce off the racket (Y, in m/sec) after having been dropped from a standard height. There were n = 10 replicates per treatment (combination of racket brand and string type). Assume these are the only levels of interest to the researchers (that is, both are fixed factors).

pA.13.a. Write out the statistical model assuming errors are iid Normal.

pA.13.b. The following table gives the sample means. Compute the following least squares estimates of model parameters and σ^2:

Means			
Racket\String	Factory(j=1)	New(j=2)	Overall
Medium Price(i=1)	0.363	0.377	0.370
High Price(i=2)	0.350	0.322	0.336
Higher Price(i=3)	0.250	0.360	0.305
Overall	0.321	0.353	0.337

$$\sum_{i=1}^{3}\sum_{j=1}^{2} s_{ij}^2 = 0.0025$$

$\hat{\mu} =$ ______ $\hat{\sigma}^2 =$ ______________________

$\hat{\alpha}_1 =$ ______________ $\hat{\alpha}_2 =$ ______________ $\hat{\alpha}_3 =$ ______________ $\hat{\beta}_1 =$ ______________ $\hat{\beta}_2 =$ ______________

$\widehat{(\alpha\beta)}_{11} =$ ______________ $\widehat{(\alpha\beta)}_{21} =$ ______________ $\widehat{(\alpha\beta)}_{31} =$ ______________

$\widehat{(\alpha\beta)}_{12} =$ ______________ $\widehat{(\alpha\beta)}_{22} =$ ______________ $\widehat{(\alpha\beta)}_{32} =$ ______________

pA.13.c. Complete the following Analysis of Variance Table, including stating the null hypothesis for each row in the table.

ANOVA							
Source	**df**	**SS**	**MS**	**F_obs**	**F(.05)**	**H_0:**	**Reject H0?**
							Yes / No
							Yes / No
							Yes / No
				#N/A	**#N/A**	**#N/A**	**#N/A**
Total				**#N/A**	**#N/A**	**#N/A**	**#N/A**

QA.14. A study was conducted to test for effects on willingness to pay during online auctions. There were 2 factors, each with 2 levels (both fixed): Urgency (Present (i=1) /Absent (i=2)) and Contrast (3 of 6 items "Featured" (High, j=1)/all 6 items "Featured" (Low, j=2)). There were 6 watches, and the response was the amount the participant was willing to pay for the watch. Although the researchers started with 80 subjects, 9 were eliminated due to incomplete information, so N = 71. The model fit is:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad i=1,2;\ j=1,2;\ k=1,...,n_{ij} \quad \alpha_1+\alpha_2=\beta_1+\beta_2=\sum_{i=1}^{2}(\alpha\beta)_{ij}=\sum_{j=1}^{2}(\alpha\beta)_{ij}=0 \quad i,j=1,2$$

pA.14.a. Give the form of the (full rank) **X** matrix and **β** vector for this model. Note that although **X** has 71 rows, there are only 4 "blocks" of distinct levels, each with a particular numbers of subjects.

pA.14.b. Obtain **X'X** as functions of the cell sample sizes (Just give the values on or above the main diagonal).

pA.14.c. The authors fit the following 4 models, with approximate Error sums of squares (divided by 1000 for ease of calculation):

Model 1: $E\{Y_{ijk}\} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$ Model 2: $E\{Y_{ijk}\} = \mu + \alpha_i + \beta_j$ Model 3: $E\{Y_{ijk}\} = \mu + \beta_j + (\alpha\beta)_{ij}$ Model 4: $E\{Y_{ijk}\} = \mu + \alpha_i + (\alpha\beta)_{ij}$

$SSErr_1 = 884.0$ $SSErr_2 = 937.0$ $SSErr_3 = 941.5$ $SSErr_4 = 978.3$

Test: $H_0^{AB}: (\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{21} = (\alpha\beta)_{22} = 0$

Test Statistic: _____________ Rejection Region: ____________ Do you conclude the Urgency effect "depends" on Contrast? Y / N

pA.14.d. The sample means for the 4 treatments are: $\overline{Y}_{11\bullet} = 216.94$ $\overline{Y}_{12\bullet} = 90.26$ $\overline{Y}_{21\bullet} = 106.12$ $\overline{Y}_{22\bullet} = 88.06$. Treating the cell sample sizes as $n_{ij} = 18$ as an approximation, use Bonferroni's method to compare all pairs of treatment means.

Part B. Mixed and Random Effects

QB.1.. The following partial ANOVA table gives the results of a balanced 2-Way ANOVA with interaction. Fill in the following values (for the mixed model, assume the unrestricted model):

Source	df	SS
A	3	600
B	5	750
AB		300
Error	72	360
Total		

$a =$ ___ $b =$ ___ $r =$ ___ $F_{AB} =$ _____ $F(.05) =$ _____ $\hat{\sigma}^2 =$ _____

A Fixed, B Fixed: $F_A =$ _____ $F(.05) =$ _____ $F_B =$ _____ $F(.05) =$ _____

A Random, B Random: $F_A =$ _____ $F(.05) =$ _____ $F_B =$ _____ $F(.05) =$ _____

A Fixed, B Random: $F_A =$ _____ $F(.05) =$ _____ $F_B =$ _____ $F(.05) =$ _____

QB.2.. For the 2-Way ANOVA with random effects,

$$y_{ijk} = \mu + a_i + b_j + e_{ijk} \quad i = 1,\ldots,a;\ j = 1,\ldots,b;\ k = 1,\ldots,r \quad a_i \sim N\left(0,\sigma_a^2\right) \quad b_j \sim N\left(0,\sigma_b^2\right) \quad e_{ijk} \sim N\left(0,\sigma^2\right) \quad \{a\} \perp \{b\} \perp \{e\}$$

Obtain the following quantities:

pB.2.a. $E\left(y_{ijk}\right)=$

pB.2.b. $COV\left(y_{ijk},y_{i'j'k'}\right)=\begin{cases} & i=i',j=j',k=k' \\ & i=i',j=j',k\neq k' \\ & i=i',j\neq j',\forall k,k' \\ & i\neq i',j=j',\forall k,k' \\ & i\neq i',j\neq j',\forall k,k' \end{cases}$

pB.2.c. $V\left(\overline{y}_{ij\bullet}\right)=$

QB.3. In a large factory, with many **O**perators, **P**arts, and **D**evices, an experiment is conducted to measure the variation in measured strengths of parts. Samples of 5 **O**perators, 10 **P**arts, and 3 **D**evices were obtained; with each combination of Operators, Parts, and Instruments being replicated 2 times. The following model is fit (with all random effects independent).

$$y_{ijkl}=\mu+o_i+p_j+d_k+op_{ij}+od_{ik}+pd_{jk}+opd_{ijk}+e_{ijkl}\quad o_i\sim N\left(0,\sigma_o^2\right)\quad p_j\sim N\left(0,\sigma_p^2\right)\quad d_k\sim N\left(0,\sigma_d^2\right)$$
$$op_{ij}\sim N\left(0,\sigma_{op}^2\right)\quad od_{ik}\sim N\left(0,\sigma_{od}^2\right)\quad pd_{jk}\sim N\left(0,\sigma_{pd}^2\right)\quad opd_{ijk}\sim N\left(0,\sigma_{opd}^2\right)\quad e_{ijkl}\sim N\left(0,\sigma^2\right)$$

pB.3.a. Complete the following ANOVA Table:

Source	df	SS	MS	E(MS)
O		800		
P		450		
D		400		
OP		720		
OD		240		
PD		90		
OPD		144		
Error				
Total		3144		

pB.3.b.i Obtain an unbiased estimate of σ_{opd}^2 ____________________________

pB.3.b.ii. Test H0: $\sigma_{opd}^2 = 0$ vs HA: $\sigma_{opd}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.3.c.i Obtain an unbiased estimate of σ_{op}^2 ___________________________

pB.3.c.ii. Test H0: $\sigma_{op}^2 = 0$ vs HA: $\sigma_{op}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.3.d.i Obtain an unbiased estimate of σ_{o}^2 ___________________________

pB.3.d.ii. Test H0: $\sigma_{o}^2 = 0$ vs HA: $\sigma_{o}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

$$\text{"Synthetic Mean Square"} = \sum_{i=1}^{c} g_i MS_i \Rightarrow \hat{\nu}^* = \frac{\left(\sum_{i=1}^{c} g_i MS_i\right)^2}{\sum_{i=1}^{c} \frac{\left(g_i MS_i\right)^2}{\nu_i}} \quad \text{where } \nu_i = df\left(MS_i\right)$$

QB.4. For the 2-way ANOVA with one fixed factor, 1 random factor and a random interaction, considering the following model:

$$y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk} \quad i = 1,\dots,a \;\; j = 1,\dots,b \;\; k = 1,\dots,r$$

$$\sum_{i=1}^{a} \alpha_i = 0 \quad b_j \sim NID\left(0,\sigma_b^2\right) \quad (ab)_{ij} \sim NID\left(0,\sigma_{ab}^2\right) \quad e_{ijk} \sim NID\left(0,\sigma^2\right) \quad \left\{b_j\right\} \perp \left\{ab_{ij}\right\} \perp \left\{e_{ijk}\right\}$$

pB.4.a. Derive $E(Y_{ij\bullet})$, $V(Y_{ij\bullet})$, $\mathrm{Cov}(Y_{ij\bullet}, Y_{ij'\bullet})$

pB.4.b. Consider the following results:

$$V\left(\bar{y}_{i\bullet\bullet}\right) = \frac{1}{br}\left(r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right) \quad COV\left(\bar{y}_{i\bullet\bullet}, \bar{y}_{i'\bullet\bullet}\right) = \frac{1}{b}\left(\sigma_b^2\right) \quad i \neq i' \quad E\left(MSAB\right) = r\sigma_{ab}^2 + \sigma^2$$

Set-up a 95% Confidence Interval for (α_i - $\alpha_{i'}$) based on these results (note that the variance components are unknown). Hint: Derive the mean and variance of $\overline{y}_{i\bullet\bullet} - \overline{y}_{i'\bullet\bullet}$ and make use of the estimated variance (standard error). Be specific (symbolically) and on degrees of freedom being used.

QB.5. The following partial ANOVA table gives the results of a balanced 2-Way ANOVA with interaction. Fill in the following values (for the mixed model, assume the unrestricted model, that is $(ab)_{ij} \sim NID(0,\sigma_{ab}^2)$):

Source	df	SS
A	2	600
B	4	820
AB		256
Error	75	
Total		2126

$$a = \_\_\_ \quad b = \_\_\_ \quad r = \_\_\_ \quad F_{AB} = \_\_\_\_\_\_ \quad F(.05) = \_\_\_\_\_ \quad \hat{\sigma}^2 = \_\_\_\_\_$$

$$A \text{ Fixed, } B \text{ Fixed: } F_A = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_ \quad F_B = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_$$

$$A \text{ Random, } B \text{ Random: } F_A = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_ \quad F_B = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_$$

$$A \text{ Fixed, } B \text{ Random: } F_A = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_ \quad F_B = \_\_\_\_\_ \quad F(.05) = \_\_\_\_\_$$

QB.6. For the 2-Way ANOVA with random effects and interactions,

$$y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk} \quad i = 1,\dots,a;\ j = 1,\dots,b;\ k = 1,\dots,r$$

$$a_i \sim NID(0,\sigma_a^2) \quad b_j \sim NID(0,\sigma_b^2) \quad (ab)_{ij} \sim NID(0,\sigma_{ab}^2) \quad e_{ijk} \sim N(0,\sigma^2) \quad \{a\} \perp \{b\} \perp \{ab\} \perp \{e\}$$

Obtain the following quantities:

pB.6.a. $E\left(y_{ijk}\right)=$

pB.6.b. $COV\left(y_{ijk}, y_{i'j'k'}\right)=\begin{cases} & i=i', j=j', k=k' \\ & i=i', j=j', k\neq k' \\ & i=i', j\neq j', \forall k,k' \\ & i\neq i', j=j', \forall k,k' \\ & i\neq i', j\neq j', \forall k,k' \end{cases}$

pB.6.c. $V\left(\overline{y}_{ij\bullet}\right)=$

QB.7. In a large factory, with many **O**perators, **P**arts, and **D**evices, an experiment is conducted to measure the variation in measured strengths of parts. Samples of 6 **O**perators, 8 **P**arts, and 4 **D**evices were obtained; with each combination of Operators, Parts, and Instruments being replicated 3 times. The following model is fit (with all random effects independent).

$$y_{ijkl}=\mu+o_i+p_j+d_k+op_{ij}+od_{ik}+pd_{jk}+opd_{ijk}+e_{ijkl} \quad o_i \sim N\left(0,\sigma_o^2\right) \quad p_j \sim N\left(0,\sigma_p^2\right) \quad d_k \sim N\left(0,\sigma_d^2\right)$$
$$op_{ij} \sim N\left(0,\sigma_{op}^2\right) \quad od_{ik} \sim N\left(0,\sigma_{od}^2\right) \quad pd_{jk} \sim N\left(0,\sigma_{pd}^2\right) \quad opd_{ijk} \sim N\left(0,\sigma_{opd}^2\right) \quad e_{ijkl} \sim N\left(0,\sigma^2\right)$$

pB.7.a. Complete the following ANOVA Table:

Source	df	SS	MS	E(MS)
O		800		
P		490		
D		420		
OP		700		
OD		240		
PD		105		
OPD		315		
Error				
Total		4030		

pB.7.b.i Obtain an unbiased estimate of σ_{opd}^2 ___________________________

pB.7.b.ii. Test H0: $\sigma_{opd}^2 = 0$ vs HA: $\sigma_{opd}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.7.c.i Obtain an unbiased estimate of σ_{op}^2 ___________________________

pB.7.c.ii. Test H0: $\sigma_{op}^2 = 0$ vs HA: $\sigma_{op}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.7.d.i Obtain an unbiased estimate of σ_o^2 ___________________________

pB.7.d.ii. Test H0: $\sigma_o^2 = 0$ vs HA: $\sigma_o^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

$$\text{"Synthetic Mean Square"} = \sum_{i=1}^{c} g_i MS_i \Rightarrow \hat{\nu}^* = \frac{\left(\sum_{i=1}^{c} g_i MS_i\right)^2}{\sum_{i=1}^{c} \frac{\left(g_i MS_i\right)^2}{\nu_i}} \quad \text{where } \nu_i = df\left(MS_i\right)$$

QB.8. A study was conducted to determine the effects of 2 map factors on subjects' abilities to complete web-map tasks. A group of 96 subjects was obtained, **and each subject was measured in each of the 4 map conditions**. Y was the time to complete the task (sec).

- Factor A: Map centering on zooming in/out (Original Center/Re-center)
- Factor B: Directional Pan (Grouped together/ Distributed on edge of map)
- Factor C: Subject (96 individuals)

pB.8.a. Write out the appropriate statistical model, making clear what effects are fixed/random.

pB.8.b. Based on the following table, obtain SSA, SSB, and SSAB.

Center\Direction	Grouped	Distributed	Mean
Original Center	107.00	113.50	110.25
Re-Center	116.80	117.90	117.35
Mean	111.90	115.70	113.80

SSA = _____________________ SSB = ______________________ SSAB = ___________________________

pB.8.c. Given the authors' results, complete the following ANOVA table and give estimates of $\sigma_c^2, \sigma_{ac}^2, \sigma_{bc}^2, \sigma_*^2 = \sigma_{abc}^2 + \sigma^2$

Source	df	SS	MS	F
Center				5.31
Direction				2.22
Cen*Dir				1.07
Subject				
Cen*Subject				
Dir*Subject				
Cen*Dir*Subject				
Total		665877.3		

$\hat{\sigma}^2_c =$ ____________ $\hat{\sigma}^2_{ac} =$ ______________ $\hat{\sigma}^2_{bc} =$ ________________ $\hat{\sigma}^2_{*} = \hat{\sigma}^2_{abc} + \hat{\sigma}^2 =$ __________________

QB.9. A mixed model is fit with 2 factors, A fixed with a=3 levels, and B random with b=levels. There are r=4 replicates per treatment (combination of levels of A and B).

pB.9.a. Write the statistical model (including main effects and interactions).

pB.9.b. Derive (showing all work):

$$E\{Y_{ij\bullet}\},\quad V\{Y_{ij\bullet}\},\quad E\{Y_{i\bullet\bullet}\},\quad V\{Y_{i\bullet\bullet}\},\quad COV\{Y_{i\bullet\bullet},Y_{i'\bullet\bullet}\}\,(i \neq i'),\quad E\{Y_{\bullet\bullet\bullet}\},\quad V\{Y_{\bullet\bullet\bullet}\},$$

QB.10. For the 2-way ANOVA with one fixed factor, 1 random factor and a random interaction, considering the following model:

$$y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk} \quad i=1,\ldots,a \;\; j=1,\ldots,b \;\; k=1,\ldots,r$$

$$\sum_{i=1}^{a}\alpha_i = 0 \quad b_j \sim NID\left(0,\sigma_b^2\right) \quad (ab)_{ij} \sim NID\left(0,\sigma_{ab}^2\right) \quad e_{ijk} \sim NID\left(0,\sigma^2\right) \quad \{b_j\} \perp \{ab_{ij}\} \perp \{e_{ijk}\}$$

pB.10.a. Derive $E(Y_{ij\bullet})$, $V(Y_{ij\bullet})$, $\mathrm{Cov}(Y_{ij\bullet}, Y_{ij'\bullet})$

pB.10.b. Consider the following results:

$$V\left(\overline{y}_{i\bullet\bullet}\right) = \frac{1}{br}\left(r\sigma_b^2 + r\sigma_{ab}^2 + \sigma^2\right) \quad COV\left(\overline{y}_{i\bullet\bullet},\overline{y}_{i'\bullet\bullet}\right) = \frac{1}{b}\left(\sigma_b^2\right) \quad i \neq i' \quad E\left(MSAB\right) = r\sigma_{ab}^2 + \sigma^2$$

Set-up a 95% Confidence Interval for ($\alpha_i - \alpha_{i'}$) based on these results (note that the variance components are unknown). Hint: Derive the mean and variance of $\overline{y}_{i\bullet\bullet} - \overline{y}_{i'\bullet\bullet}$ and make use of the estimated variance (standard error). Be specific (symbolically) and on degrees of freedom being used.

QB.11. For the 2-Way Random Effects model, with a = 2, b = 4, and r = 3; DERIVE the following Variances:

$$y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk} \quad i = 1,2;\ j = 1,2,3,4;\ k = 1,2,3$$
$$a_i \sim NID\left(0,\sigma_a^2\right) \quad b_j \sim NID\left(0,\sigma_b^2\right) \quad (ab)_{ij} \sim NID\left(0,\sigma_{ab}^2\right) \quad e_{ijk} \sim N\left(0,\sigma^2\right) \quad \{a\} \perp \{b\} \perp \{ab\} \perp \{e\}$$

$$V\{y_{ijk}\},\quad V\{\overline{y}_{ij\bullet}\},\quad V\{\overline{y}_{i\bullet\bullet}\},\quad V\{\overline{y}_{\bullet j\bullet}\},\quad V\{\overline{y}_{\bullet\bullet\bullet}\}$$

QB.11. A study is conducted to measure intra- and inter-observer reliability in tasting experiments. A random sample of *a* = 10 **V**arieties of wine were selected at a wine store. Further, a random sample of *b* = 5 trained **J**udges were obtained from a society of wine aficionados. Each judge rated each wine *r* = 3 times. The judge gave the wine an evaluation for a particular attribute on a visual analogue scale from 0-100. (Note: The order of the 30 tastes for each judge were randomized and blinded, and the judges were given cab rides home). The model fit is:

$$y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk} \quad i = 1,\dots,a;\ j = 1,\dots,b;\ k = 1,\dots,r$$
$$a_i \sim NID\left(0,\sigma_V^2\right) \quad b_j \sim NID\left(0,\sigma_J^2\right) \quad (ab)_{ij} \sim NID\left(0,\sigma_{VJ}^2\right) \quad e_{ijk} \sim N\left(0,\sigma^2\right) \quad \{a\} \perp \{b\} \perp \{ab\} \perp \{e\}$$

pB.11.a. Complete the following ANOVA Table and give the expected mean square for each row. Note that for the F-Stats, you are testing: $H_0^V : \sigma_V^2 = 0 \quad H_A^V : \sigma_V^2 > 0 \qquad H_0^J : \sigma_J^2 = 0 \quad H_A^J : \sigma_J^2 > 0 \qquad H_0^{VJ} : \sigma_{VJ}^2 = 0 \quad H_A^{VJ} : \sigma_{VJ}^2 > 0$

Source	**df**	**SS**	**MS**	**F-Stat**	**F(.05)**	**E(MS)**
Wine Variety (V)		**8100**				
Judge (J)		**2000**				
Variety x Judge (VJ)		**1440**				
Error						
Total		**12740**				

pB.11.b. Give unbiased (ANOVA) estimates of each of the four variance components:

$\hat{\sigma}_V^2 =$ _______________ $\hat{\sigma}_J^2 =$ _______________ $\hat{\sigma}_{VJ}^2 =$ _______________ $\hat{\sigma}^2 =$ _______________

pB.11.c. For this type of reliability study, there are two types of correlations: Inter-Judge (Among), and Intra-Judge (Within). The Inter-Judge and Intra-Judge Covariances, as well as Observation measurement Variance are used to obtain the two correlations. Give these terms (based on actual variance components) and their point estimates:

Measurement: $V\left\{Y_{ijk}\right\} =$ ________________________ $\hat{V}\left\{Y_{ijk}\right\} =$ ________________________

Inter-Judge: $COV\left\{Y_{ijk}, Y_{ij'k'}\right\} =$ ________________ $j \neq j' \quad \forall\, k, k'$ $\hat{COV}\left\{Y_{ijk}, Y_{ij'k'}\right\} =$ ________________

Intra-Judge: $COV\left\{Y_{ijk}, Y_{ijk'}\right\} =$ ________________________ $k \neq k'$ $\hat{COV}\left\{Y_{ijk}, Y_{ijk'}\right\} =$ ____________________

pB.11.d. The Inter-Judge and Intra-Judge Correlations are defined below, give “simple” point estimates of each:

$$ICC_{\text{Inter}} = \frac{COV\left\{Y_{ijk}, Y_{ij'k'}\right\}}{V\left\{Y_{ijk}\right\}} \qquad \hat{ICC}_{\text{Inter}} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$ICC_{\text{Intra}} = \frac{COV\left\{Y_{ijk}, Y_{ijk'}\right\}}{V\left\{Y_{ijk}\right\}} \qquad \hat{ICC}_{\text{Intra}} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

QB.12. In a large factory, with many **O**perators, **P**arts, and **D**evices, an experiment is conducted to measure the variation in measured strengths of parts. Samples of 4 **O**perators, 10 **P**arts, and 3 **D**evices were obtained; with each combination of Operators, Parts, and Instruments being replicated 5 times. The following model is fit (with all random effects independent).

$$y_{ijkl} = \mu + o_i + p_j + d_k + op_{ij} + od_{ik} + pd_{jk} + opd_{ijk} + e_{ijkl} \quad o_i \sim N\left(0, \sigma_o^2\right) \quad p_j \sim N\left(0, \sigma_p^2\right) \quad d_k \sim N\left(0, \sigma_d^2\right)$$
$$op_{ij} \sim N\left(0, \sigma_{op}^2\right) \quad od_{ik} \sim N\left(0, \sigma_{od}^2\right) \quad pd_{jk} \sim N\left(0, \sigma_{pd}^2\right) \quad opd_{ijk} \sim N\left(0, \sigma_{opd}^2\right) \quad e_{ijkl} \sim N\left(0, \sigma^2\right)$$

pB.12.a. Complete the following ANOVA Table:

Source	df	SS	MS	E(MS)
O		1200		
P		450		
D		160		
OP		540		
OD		60		
PD		144		
OPD		270		
Error				
Total		5224		

pB.12.b.i Obtain an unbiased estimate of σ_{opd}^2 ___________________________

pB.12.b.ii. Test H_0: $\sigma_{opd}^2 = 0$ vs H_A: $\sigma_{opd}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.12.c.i Obtain an unbiased estimate of σ_{op}^2 ___________________________

pB.12.c.ii. Test H_0: $\sigma_{op}^2 = 0$ vs H_A: $\sigma_{op}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

pB.12.d.i Obtain an unbiased estimate of σ_{o}^2 ___________________________

pB.12.d.ii. Test H_0: $\sigma_{o}^2 = 0$ vs H_A: $\sigma_{o}^2 > 0$ Test Statistic: ______________ Rejection Region: ____________

$$\text{"Synthetic Mean Square"} = \sum_{i=1}^{c} g_i MS_i \Rightarrow \hat{\nu}^* = \frac{\left(\sum_{i=1}^{c} g_i MS_i\right)^2}{\sum_{i=1}^{c} \frac{\left(g_i MS_i\right)^2}{\nu_i}} \quad \text{where } \nu_i = df\left(MS_i\right)$$

QB.13. A 2-Way Random Effects model is fit, where a sample of *a* = 8 products were measured by a sample of *b* = 6 machinists, with *n* = 3 replicates per machinist per product. The model fit is as follows (independent random effects):

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad \alpha_i \sim NID\left(0,\sigma_\alpha^2\right) \quad \beta_j \sim NID\left(0,\sigma_\beta^2\right) \quad (\alpha\beta)_{ij} \sim NID\left(0,\sigma_{\beta}^2\right) \quad \varepsilon_{ijk} \sim NID\left(0,\sigma^2\right)$$

You are given the following sums of squares: $SS_A = 420 \quad SS_B = 350 \quad SS_{AB} = 140 \quad SS_{\text{Err}} = 210$

Give the test statistic and rejection region for the following 3 tests. Note for test 1, your rejection region will be symbolic, give the specific numerator and denominator degrees of freedom. Also give unbiased (ANOVA) estimates of each variance component.

1) $H_0^{AB}: \sigma_{\alpha\beta}^2 = 0 \quad H_A^{AB}: \sigma_{\alpha\beta}^2 > 0$ 2) $H_0^{A}: \sigma_{\alpha}^2 = 0 \quad H_A^{A}: \sigma_{\alpha}^2 > 0$ 3) $H_0^{B}: \sigma_{\beta}^2 = 0 \quad H_A^{B}: \sigma_{\beta}^2 > 0$

1: Test Stat: ______________________________ Rejection Region: ___________________ Estimate: __________________

2: Test Stat: ______________________________ Rejection Region: ___________________ Estimate: __________________

3: Test Stat: ______________________________ Rejection Region: ___________________ Estimate: __________________

QB14. Mixed model with factor A fixed with a = 2 levels and factor B random with 3 levels, random interaction, and n = 2 replicates per combination. Give the model in terms of **Y = Xβ + Zu + ε**. Give all elements in full form.

QB.15. An experiment is conducted to estimate the effects of 3 brands of shoes on one mile run times for a sample of 4 runners. Each runner's time is measured 5 times in each brand of shoe. The experimenters treat the shoes as fixed and the runners as random.

pB.15.a. Write out the statistical model, allowing for main effects and interaction of shoes and runners (be specific on parameters and ranges of subscripts):

pB.15.b. Complete the following table. Whenever possible, use actual numbers, not symbols. For the sum of squares, write out the relevant summation.

Source	df	Sum of Squares	E{MS}

QB.16. A wine tasting is conducted, with a sample of *a* judges, each rating a sample of *b* wines. Note that each judge only rates each wine once.

$$y_{ij} = \mu + a_i + b_j + e_{ij} \quad i=1,\ldots,a;\ j=1,\ldots,b \quad a_i \sim NID\left(0,\sigma_a^2\right) \quad b_j \sim NID\left(0,\sigma_b^2\right) \quad e_{ij} \sim NID\left(0,\sigma^2\right) \quad \{a_i\} \perp \{b_j\} \perp \{e_{ij}\}$$

pB.16.a. Showing all work, obtain: $E\{y_{ij}\}, V\{y_{ij}\}, E\{\overline{y}_{i\bullet}\}, V\{\overline{y}_{i\bullet}\}, E\{\overline{y}_{\bullet j}\}, V\{\overline{y}_{\bullet j}\}, E\{\overline{y}_{\bullet\bullet}\}, V\{\overline{y}_{\bullet\bullet}\}$

pB.16.b. Derive $E\{MSA\} = E\left\{\frac{1}{a-1}\left[b\sum_{i=1}^{a}\overline{y}_{i\bullet}^{2} - ab\overline{y}_{\bullet\bullet}^{2}\right]\right\}$

pB.16.c. The study had 9 judges and 10 varieties of wine. Complete the following ANOVA table testing:

$$H_0^A:\sigma_A^2 = 0 \quad H_A^A:\sigma_A^2 > 0 \quad H_0^B:\sigma_B^2 = 0 \quad H_A^B:\sigma_B^2 > 0$$

Source	df	SS	MS	F	F(0.05)
Judge		231.07			
Variety		23.89			
Error		324.98			
Total		579.95			

pB.16.d. Obtain unbiased estimates of σ_A^2 and σ^2.

QB.17. An experiment is conducted to compare 4 navigation techniques (Factor A, Fixed), 2 input methods (Factor B, Fixed), in 36 subjects (Factor C, Random). Each subject is measured in each combination of levels of A and B once. Consider the model, where y is the task completion time:

$$y_{ijk} = \mu + \alpha_i + \beta_j + c_k + (\alpha\beta)_{ij} + (ac)_{ik} + (bc)_{jk} + e_{ijk} \quad \sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = \sum_{i=1}^{a}(\alpha\beta)_{ij} = \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$$

$$c_k \sim NID\left(0,\sigma_c^2\right) \quad (ac)_{ik} \sim NID\left(0,\sigma_{ac}^2\right) \quad (bc)_{jk} \sim NID\left(0,\sigma_{bc}^2\right) \quad e_{ijk} \sim NID\left(0,\sigma^2\right) \quad \{c_k\} \perp \{(ac)_{ik}\} \perp \{(bc)_{jk}\} \perp \{e_{ijk}\}$$

Complete the following ANOVA table, testing all main effects and 2-way interactions.

Satterthwaite's Approximation: $W = \sum_i g_i MS_i \quad \Rightarrow \quad df_W \approx \frac{\left(\sum_i g_i MS_i\right)^2}{\sum_i \frac{(g_i MS_i)^2}{df_i}}$

ANOVA							
Source	df	SS	MS	F	df_num	df_den	F(0.05)
A		66996					
B		30636					
C		84008					
AB		18710					
AC		148797					
BC		68605					
Error		97282					
Total		515034					

QB.18. A study was conducted, regarding 2 Random factors: A: Wine Judge (a = 9) and B: Wine Brand (b = 10). Each judge rated each brand once, blind to the brand label. The model fit is:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \quad i = 1,\ldots,9;\ j = 1,\ldots,10 \quad \alpha_i \sim NID\left(0,\sigma_\alpha^2\right) \quad \beta_j \sim NID\left(0,\sigma_\beta^2\right) \quad \varepsilon_{ij} \sim NID\left(0,\sigma^2\right) \quad \{\alpha\} \perp \{\beta\} \perp \{\varepsilon\}$$

pB.18.a. For this model,

$$SS_{\text{ERR}} = \sum_i \sum_j \left(Y_{ij} - \overline{Y}_{i\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet}\right)^2 = \sum_i \sum_j Y_{ij}^2 - b\sum_i \overline{Y}_{i\bullet}^2 - a\sum_j \overline{Y}_{\bullet j}^2 + ab\overline{Y}_{\bullet\bullet}^2 \qquad MS_{\text{ERR}} = \frac{SS_{\text{ERR}}}{(a-1)(b-1)}$$

pB.18.a.i.

$E\left\{Y_{ij}\right\} =$ ______________ $E\left\{\overline{Y}_{i\bullet}\right\} =$ ______________ $E\left\{\overline{Y}_{\bullet j}\right\} =$ ______________ $E\left\{\overline{Y}_{\bullet\bullet}\right\} =$ ______________

pB.18.a.ii. **Derive** $V\left\{Y_{ij}\right\},\ V\left\{\overline{Y}_{i\bullet}\right\},\ V\left\{\overline{Y}_{\bullet j}\right\},\ V\left\{\overline{Y}_{\bullet\bullet}\right\}$ **Be very specific on all parts of derivation.**

pB.18.b. Use the results from above to obtain $E\left\{MS_{\text{ERR}}\right\}$

pB.18.c. Write out MS_A in terms of (a subset) of the components in SS_{ERR} and obtain E{MS_A}.

pB.18.d. Write out MS_B in terms of (a subset) of the components in SS_{ERR} and obtain E{MS_B}.

pB.18.e. From the published study, we obtain the following sums of squares:

$$SS_A = 180.32 \quad SS_B = 114.90 \quad SS_{\text{ERR}} = 395.12$$

Obtain unbiased estimates of the model's three variances.

pB.18.f. Out of the (estimated) total variance $V\left\{Y_{ijk}\right\} = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma^2$, what percentages are accounted by the 3 sources:

Judges (A) ____________________ Brands (B) ____________________ Error (AB) ____________________

QB.19. Based on the 2014 WNBA season, we have the point totals (Y) by game **Location** (Home/Away) for a **sample** of 10 **Players**. Each player played 17 home games and 17 away games. Consider the model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,a \quad j = 1,\ldots,b \quad k = 1,\ldots,n \quad \sum_{i=1}^{a} \alpha_i = 0 \quad \beta_j \sim N\left(0,\sigma_\beta^2\right) \quad \alpha\beta_{ij} \sim N\left(0,\sigma_{\alpha\beta}^2\right) \quad \varepsilon_{ijk} \sim N\left(0,\sigma^2\right)$$

ANOVA					
Source	df	SS	MS	F	F(0.05)
Player		3879.30			
Location		1.30			
P*L		323.67			
Error		15787.29		#N/A	#N/A
Total		19991.56	#N/A	#N/A	#N/A

pB.19.a. Complete the partial ANOVA table.

pB.19.b. Test whether there is an interaction between Player and Location (Home). $H_0: \sigma_{\alpha\beta}^2 = 0$

pB.19.b.i. Test Stat: _______ p.5.b.ii. Reject H_0 if Test Stat is in the range _________ p.5.b.iii. P-value **>** **or** **<** .05?

pB.19.c. Test whether there is Location (Home vs Away) Main Effect. $H_0: \alpha_1 = \alpha_2 = 0$

pB.19.c.i. Test Stat: _______ p.5.c.ii. Reject H_0 if Test Stat is in the range _________ p.5.c.iii. P-value **>** **or** **<** .05?

pB.19.d. Test whether there is Player Main Effect. $H_0: \sigma_\beta^2 = 0$

pB.19.d.i. Test Stat: _______ p.5.d.ii. Reject H_0 if Test Stat is in the range _________ p.5.d.iii. P-value **>** **or** **<** .05?

pB.19.e. Give unbiased estimates of each of the variance components:

$\hat{\sigma}_{\alpha\beta}^2 =$ ________________________ $\hat{\sigma}_\beta^2 =$ ________________________ $\hat{\sigma}^2 =$ ________________________