Experimental Design and the Analysis of Variance

Comparing *t* > 2 Groups - Numeric Responses

- Extension of Methods used to Compare 2 Groups
- Independent Samples and Paired Data Designs
- Normal and non-normal data distributions

Data Design	Normal	Non-normal
Independent Samples (CRD)	F-Test 1-Way ANOVA	Kruskal-Wallis Test
Paired Data (RBD)	F-Test 2-Way ANOVA	Friedman's Test

Completely Randomized Design (CRD)

- Controlled Experiments - Subjects assigned at random to one of the *t* treatments to be compared
- Observational Studies - Subjects are sampled from *t* existing groups
- Statistical model y_{ij} is measurement from the j^{th} subject from group *i*:

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} = \mu_i + \varepsilon_{ij}$$

where μ is the overall mean, α_i is the effect of treatment *i* , ε_{ij} is a random error, and μ_i is the population mean for group *i*

1-Way ANOVA for Normal Data (CRD)

- For each group obtain the mean, standard deviation, and sample size:

$$\overline{y}_{i.} = \frac{\sum_j y_{ij}}{n_i} \qquad s_i = \sqrt{\frac{\sum_j (y_{ij} - \overline{y}_{i.})^2}{n_i - 1}}$$

- Obtain the overall mean and sample size

$$N = n_1 + \ldots + n_t \qquad \overline{y}_{..} = \frac{n_1 \overline{y}_{1.} + \ldots + n_t \overline{y}_{t.}}{N} = \frac{\sum_i \sum_j y_{ij}}{N}$$

Analysis of Variance - Sums of Squares

- Total Variation

$$TSS = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (y_{ij} - \overline{y}_{..})^2 \qquad df_{Total} = N - 1$$

- Between Group (Sample) Variation

$$SST = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (\overline{y}_{i.} - \overline{y}_{..})^2 = \sum_{i=1}^{t} n_i (\overline{y}_{i.} - \overline{y}_{..})^2 \qquad df_T = t - 1$$

- Within Group (Sample) Variation

$$SSE = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (y_{ij} - \overline{y}_{i.})^2 = \sum_{i=1}^{t} (n_i - 1) s_i^2 \qquad df_E = N - t$$

$$TSS = SST + SSE \qquad df_{Total} = df_T + df_E$$

Analysis of Variance Table and *F*-Test

Source of Variation	Sum of Squares	Degrres of Freedom	Mean Square	*F*
Treatments	*SST*	t-1	$MST=SST/(t-1)$	$F=MST/MSE$
Error	*SSE*	N-t	$MSE=SSE/(N-t)$	
Total	*TSS*	N-1		

- Assumption: All distributions normal with common variance
- H_0: No differences among Group Means ($\alpha_1 = \cdots = \alpha_t = 0$)
- H_A: Group means are not all equal (Not all α_i are 0)

$$T.S.: F_{obs} = \frac{MST}{MSE}$$

$$R.R.: F_{obs} \geq F_{\alpha, t-1, N-t} \qquad (Table\ 9)$$

$$P-val: P(F \geq F_{obs})$$

Expected Mean Squares

- Model: $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ with $\varepsilon_{ij} \sim N(0,\sigma^2)$, $\Sigma n_i a_i = 0$:

$$E(MSE) = \sigma^2$$

$$E(MST) = \sigma^2 + \frac{\sum_{i=1}^{t} n_i \alpha_i^2}{t-1}$$

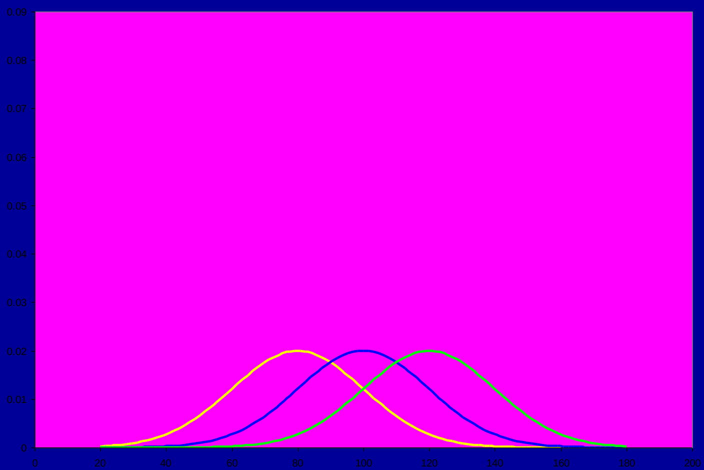
$$\Rightarrow \frac{E(MST)}{E(MSE)} = \frac{\sigma^2 + \frac{\sum_{i=1}^{t} n_i \alpha_i^2}{t-1}}{\sigma^2} = 1 + \frac{\sum_{i=1}^{t} n_i \alpha_i^2}{\sigma^2 (t-1)}$$

When $H_0 : \alpha_1 = \cdots = \alpha_t = 0$ is true, $\frac{E(MST)}{E(MSE)} = 1$

otherwise (H_a is true), $\frac{E(MST)}{E(MSE)} > 1$

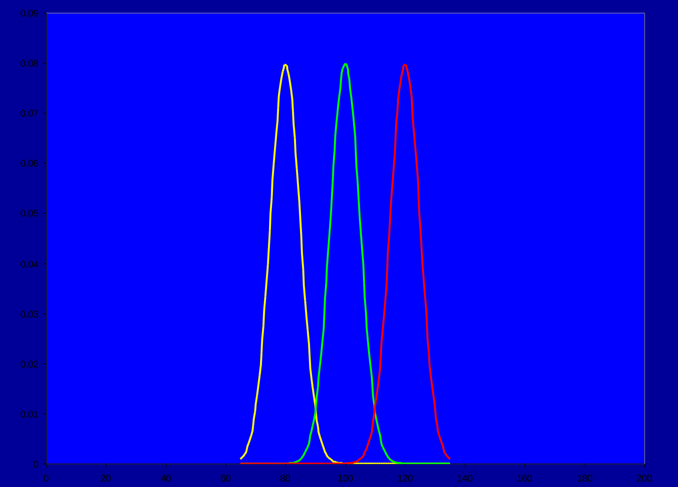
Expected Mean Squares

- 3 Factors effect magnitude of F-statistic (for fixed t)
  - True group effects ($\alpha_1,\ldots,\alpha_t$)
  - Group sample sizes ($n_1,\ldots,n_t$)
  - Within group variance (σ^2)
- $F_{obs} = MST/MSE$
- When H_0 is true ($\alpha_1=\ldots=\alpha_t=0$), $E(MST)/E(MSE)=1$
- Marginal Effects of each factor (all other factors fixed)
  - As spread in ($\alpha_1,\ldots,\alpha_t$) $\uparrow$ $E(MST)/E(MSE) \uparrow$
  - As ($n_1,\ldots,n_t$) $\uparrow$ $E(MST)/E(MSE) \uparrow$ (when H_0 false)
  - As $\sigma^2 \uparrow$ $E(MST)/E(MSE) \downarrow$ (when H_0 false)



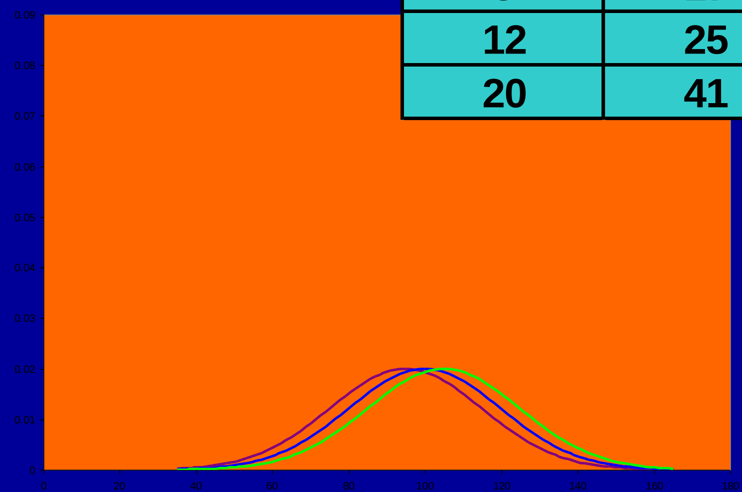
A) μ=100, τ_1=-20, τ_2=0, τ_3=20, σ = 20

$$\frac{E(MST)}{E(MSE)}$$

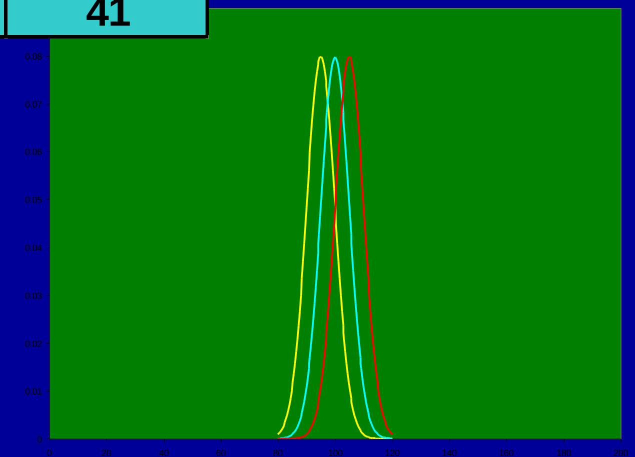


B) μ=100, τ_1=-20, τ_2=0, τ_3=20, σ = 5

n	A	B	C	D
4	9	129	1.5	9
8	17	257	2	17
12	25	385	2.5	25
20	41	641	3.5	41



C) μ=100, τ_1=-5, τ_2=0, τ_3=5, σ = 20



D) μ=100, τ_1=-5, τ_2=0, τ_3=5, σ = 5

Example - Seasonal Diet Patterns in Ravens

- "Treatments" - t = 4 seasons of year (3 "replicates" each)
  - Winter: November, December, January
  - Spring: February, March, April
  - Summer: May, June, July
  - Fall: August, September, October
- Response (Y) - Vegetation (percent of total pellet weight)
- Transformation (For approximate normality):

$$Y' = \arcsin\left(\sqrt{\frac{Y}{100}}\right)$$

Source: K.A. Engel and L.S. Young (1989). "Spatial and Temporal Patterns in the Diet of Common Ravens in Southwestern Idaho," *The Condor*, 91:372-378

Seasonal Diet Patterns in Ravens - Data/Means

Y	Winter(i=1)	Fall(i=2)	Summer(i=3)	Fall (i=4)
j=1	94.3	80.7	80.5	67.8
j=2	90.3	90.5	74.3	91.8
j=3	83.0	91.8	32.4	89.3

Y'	Winter(i=1)	Fall(i=2)	Summer(i=3)	Fall (i=4)
j=1	1.329721	1.115957	1.113428	0.967390
j=2	1.254080	1.257474	1.039152	1.280374
j=3	1.145808	1.280374	0.605545	1.237554

$$\bar{y}_{1.} = \frac{1.329721 + 1.254080 + 1.145808}{3} = 1.24203$$

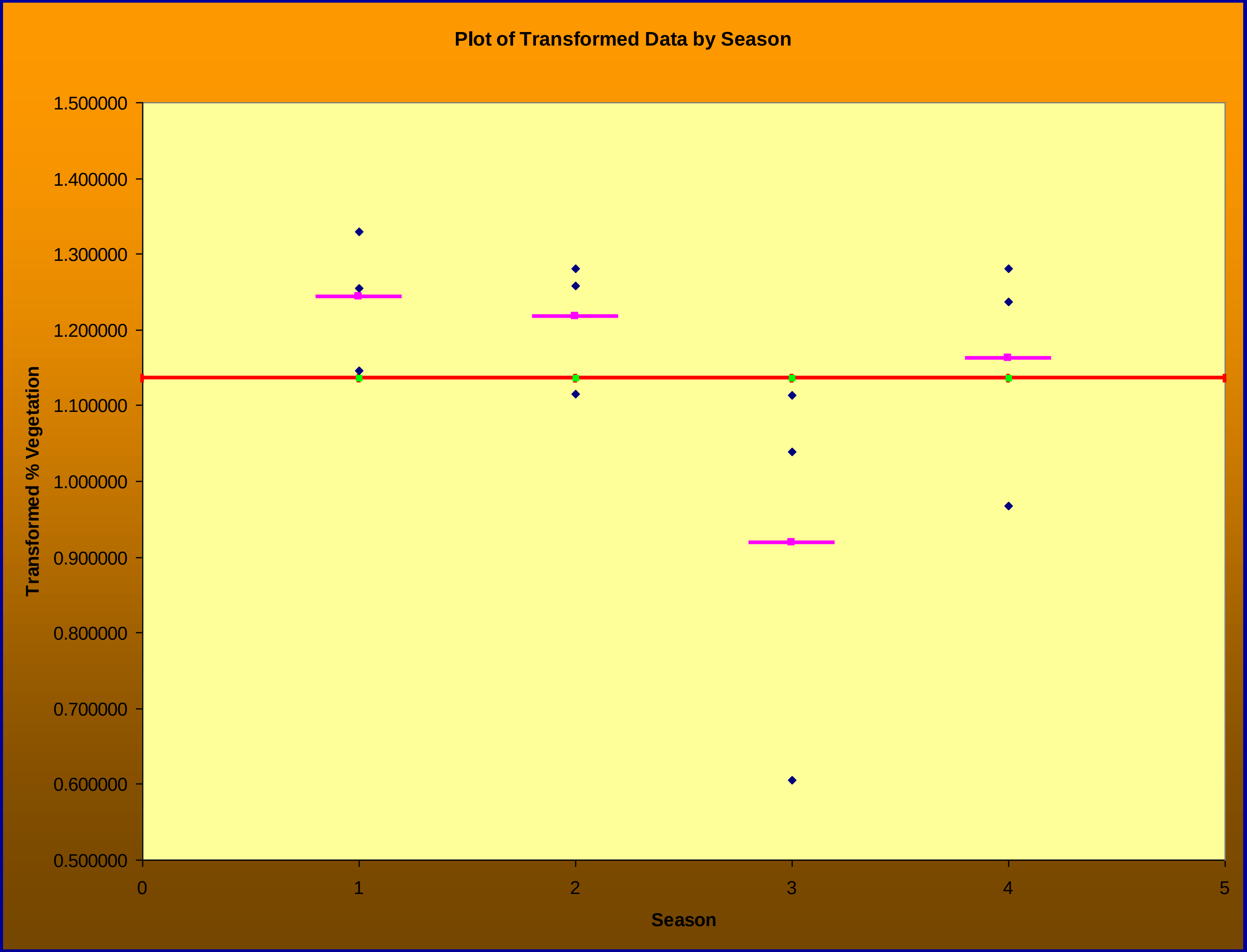
$$\bar{y}_{2.} = \frac{1.115957 + 1.257474 + 1.280374}{3} = 1.217935$$

$$\bar{y}_{3.} = \frac{1.113428 + 1.039152 + 0.605545}{3} = 0.919375$$

$$\bar{y}_{4.} = \frac{0.967390 + 1.280374 + 1.237554}{3} = 1.16773$$

$$\bar{y}_{..} = \frac{1.329721 + \ldots + 1.237554}{12} = 1.135572$$

Seasonal Diet Patterns in Ravens - Data/Means



Seasonal Diet Patterns in Ravens - ANOVA

Total Variation : ($df_{Total} = 12 - 1 = 11$)

$$TSS = (1.329721 - 1.135572)^2 + ... + (1.27554 - 1.135572)^2 = 0.438425$$

Between Group Variation : ($df_T = 4 - 1 = 3$)

$$SST = 3\left[(1.24203 - 1.135572)^2 + ... + (1.161773 - 1.135572)^2\right] = 0.197387$$

Within Group Variation : ($df_E = 12 - 4 = 8$)

$$SSE = (1.329721 - 1.243203)^2 + ... + (1.237554 - 1.161773)^2 = 0.241038$$

ANOVA						
Source of Variation	***SS***	***df***	***MS***	***F***	***P-value***	***F crit***
Between Groups	**0.197387**	**3**	**0.065796**	**2.183752**	**0.167768**	**4.06618**
Within Groups	**0.241038**	**8**	**0.03013**			
Total	**0.438425**	**11**				

Do not conclude that seasons differ with respect to vegetation intake

Seasonal Diet Patterns in Ravens - Spreadsheet

Month	Season	Y'	Season Mean	Overall Mean	TSS	SST	SSE
NOV	1	1.329721	1.243203	1.135572	0.037694	0.011584	0.007485
DEC	1	1.254080	1.243203	1.135572	0.014044	0.011584	0.000118
JAN	1	1.145808	1.243203	1.135572	0.000105	0.011584	0.009486
FEB	2	1.115957	1.217935	1.135572	0.000385	0.006784	0.010400
MAR	2	1.257474	1.217935	1.135572	0.014860	0.006784	0.001563
APR	2	1.280374	1.217935	1.135572	0.020968	0.006784	0.003899
MAY	3	1.113428	0.919375	1.135572	0.000490	0.046741	0.037657
JUN	3	1.039152	0.919375	1.135572	0.009297	0.046741	0.014346
JUL	3	0.605545	0.919375	1.135572	0.280928	0.046741	0.098489
AUG	4	0.967390	1.161773	1.135572	0.028285	0.000687	0.037785
SEP	4	1.280374	1.161773	1.135572	0.020968	0.000687	0.014066
OCT	4	1.237554	1.161773	1.135572	0.010400	0.000687	0.005743
				Sum	0.438425	0.197387	0.241038

Total SS — (Y'-Overall Mean)^2

Between Season SS — (Group Mean-Overall Mean)^2

Within Season SS — (Y'-Group Mean)^2

CRD with Non-Normal Data
Kruskal-Wallis Test

- Extension of Wilcoxon Rank-Sum Test to $t > 2$ Groups
- Procedure:
  - Rank the observations across groups from smallest (1) to largest ($N = n_1+...+n_t$), adjusting for ties
  - Compute the rank sums for each group: $T_1,...,T_t$. Note that $T_1+...+T_t = N(N+1)/2$

Kruskal-Wallis Test

- H_0: The t population distributions are identical ($M_1 = \ldots = M_t$)
- H_A: Not all t distributions are identical (Not all M_i are equal)

$$T.S.: H = \frac{12}{N(N+1)} \sum_{i=1}^{t} \frac{T_i^2}{n_i} - 3(N+1)$$

$$R.R.: H \geq \chi^2_{\alpha, t-1}$$

$$P - val: P(\chi^2 \geq H)$$

An adjustment to H is suggested when there are many ties in the data. Formula is given on page 344 of O&L.

Example - Seasonal Diet Patterns in Ravens

Month	Season	Y'	Rank
NOV	1	1.329721	12
DEC	1	1.254080	8
JAN	1	1.145808	6
FEB	2	1.115957	5
MAR	2	1.257474	9
APR	2	1.280374	10.5
MAY	3	1.113428	4
JUN	3	1.039152	3
JUL	3	0.605545	1
AUG	4	0.967390	2
SEP	4	1.280374	10.5
OCT	4	1.237554	7

- $T_1 = 12+8+6 = 26$
- $T_2 = 5+9+10.5 = 24.5$
- $T_3 = 4+3+1 = 8$
- $T_4 = 2+10.5+7 = 19.5$

H_0 : No seasonal difference $\quad$ H_a : Seasonal Differences

$$T.S.: H = \frac{12}{12(12+1)}\left[\frac{(26)^2}{3} + \frac{(24.5)^2}{3} + \frac{(8)^2}{3} + \frac{(19.5)^2}{3}\right] - 3(12+1) = 44.12 - 39 = 5.12$$

$$R.R.(\alpha = 0.05): H \geq \chi^2_{.05,4-1} = 7.815$$

$$P-\text{value}: P(\chi^2 \geq H = 5.12) = .1632$$

Transformations for Constant Variance

$\sigma^2 = k\mu$ $\quad y_T = \sqrt{y}$ or $y_T = \sqrt{y + 0.375}$ $\quad$ Used for Poisson Distribution $(k = 1)$

$\sigma^2 = k\mu^2$ $\quad y_T = \ln(y)$ or $y_T = \ln(y+1)$

$\sigma^2 = k\pi(1-\pi)$ $\quad y_T = \sin^{-1}\left(\sqrt{y}\right)$ $\quad$ Used for Binomial Distrbution $\left(k = \frac{1}{n}, \quad y = \hat{\pi}\right)$

Box-Cox Transformation: Power Transformation used to obtain normality and often constant variance

$$y_T = \begin{cases} y^{\lambda} & \lambda \neq 0 \\ \ln(y) & \lambda = 0 \end{cases}$$

Welch's Test – Unequal Variances

$$w_i = \frac{n_i}{s_i^2} \qquad w_\bullet = \sum_{i=1}^{t} w_i$$

$$F^* = \frac{1}{t-1}\left[\sum_{i=1}^{t} w_i \bar{y}_i^2 - \frac{\left(\sum_{i=1}^{t} w_i \bar{y}_i\right)^2}{w_\bullet}\right]$$

$$C_W = \sum_{i=1}^{t}\left[\frac{1}{n_i - 1}\left(1 - \frac{w_i}{w_\bullet}\right)^2\right] \qquad m_W = \left[1 + \frac{2(t-2)}{t^2-1} C_W\right]^{-1}$$

$$\nu_W = \left[\frac{3}{t^2-1} C_W\right]^{-1} \quad \Rightarrow \quad F_W = m_W F^* \overset{\text{approx}}{\sim} F_{t-1,\nu_W}$$

Example – Seasonal Diet Patterns in Ravens

Season	n_i	s_i	ybar_i	s_i^2	w_i	w_iyb_i	w_iyb_i^2	C_Wi
1	3	0.092438	1.243203	0.008545	351.091	436.4773	542.6298	0.178816
2	3	0.089055	1.217935	0.007931	378.275	460.7145	561.1204	0.160688
3	3	0.274311	0.919375	0.075246	39.86905	36.6546	33.69932	0.455394
4	3	0.169696	1.161773	0.028797	104.1779	121.0311	140.6106	0.387837
Sum					873.4129	1054.877	1278.06	1.182735

$$F^* = \frac{1}{t-1}\left[\sum_{i=1}^{t} w_i \bar{y}_i^{2} - \frac{\left(\sum_{i=1}^{t} w_i \bar{y}_i\right)^2}{w_\bullet}\right] = \frac{1}{4-1}\left[1278.06 - \frac{(1054.877)^2}{873.4129}\right] = 1.3390$$

$$C_W = \sum_{i=1}^{t}\left[\frac{1}{n_i-1}\left(1-\frac{w_i}{w_\bullet}\right)^2\right] = 1.1827 \qquad {}^{-1}$$

$$m_W = \left[1+\frac{2(t-2)}{t^2-1}C_W\right] = \left[1+\frac{2(4-2)}{4^2-1}1.1827\right]^{-1} = 0.7602$$

$$\nu_W = \left[\frac{3}{t^2-1}C_W\right]^{-1} = \left[\frac{3}{4^2-1}1.1827\right]^{-1} = 4.23 \quad \Rightarrow \quad F_W = m_W F^* = 0.7602(1.3390) = 1.0179$$

$$F_{0.05,3,4.23} = 6.4231 \quad \left(\text{qf}(.95, 3, 4.23) \text{ in R}\right)$$

Linear Contrasts

- Linear functions of the treatment means (population and sample) such that the coefficients sum to 0.
- Used to compare groups or pairs of treatment means, based on research question(s) of interest

Population Contrast: $l = a_1\mu_1 + \ldots + a_t\mu_t = \sum_{i=1}^{t} a_i\mu_i$ where $\sum_{i=1}^{t} a_i = 0$

Estimated Contrast: $\hat{l} = a_1\overline{y}_{1\bullet} + \ldots + a_t\overline{y}_{t\bullet} = \sum_{i=1}^{t} a_i\overline{y}_{i\bullet}$

$$V\left\{\hat{l}\right\} = a_1^2\left(\frac{\sigma^2}{n_1}\right) + \ldots + a_t^2\left(\frac{\sigma^2}{n_t}\right) = \sigma^2\sum_{i=1}^{t}\frac{a_i^2}{n_i} \qquad \hat{V}\left\{\hat{l}\right\} = MSE\sum_{i=1}^{t}\frac{a_i^2}{n_i} \qquad \hat{SE}\left\{\hat{l}\right\} = \sqrt{MSE\sum_{i=1}^{t}\frac{a_i^2}{n_i}}$$

$$n_1 = \ldots = n_t = n \quad \Rightarrow \quad \hat{V}\left\{\hat{l}\right\} = \frac{MSE}{n}\sum_{i=1}^{t}a_i^2 \qquad \hat{SE}\left\{\hat{l}\right\} = \sqrt{\frac{MSE}{n}\sum_{i=1}^{t}a_i^2}$$

Orthogonal Contrasts & Sums of Squares

Two Contrasts: $l_1 = \sum_{i=1}^{t} a_i \mu_i \quad l_2 = \sum_{i=1}^{t} b_i \mu_i \quad \sum_{i=1}^{t} a_i = \sum_{i=1}^{t} b_i = 0$

l_1, l_2 are orthogonal if: $\sum_{i=1}^{t} \frac{a_i b_i}{n_i} = 0$ for balanced data, $\sum_{i=1}^{t} a_i b_i = 0$

Contrast Sum of Squares: $SSC = \frac{\left(\sum_{i=1}^{t} a_i \overline{y}_{i\bullet}\right)^2}{\sum_{i=1}^{t} \frac{a_i^2}{n_i}} = \frac{\left(\hat{l}\right)^2}{\sum_{i=1}^{t} \frac{a_i^2}{n_i}}$ for balanced data, $SSC = \frac{n\left(\hat{l}\right)^2}{\sum_{i=1}^{t} a_i^2}$

Among t treatments, we can obtain $t-1$ pairwise orthogonal Contrasts: $l_1, \ldots, l_{t-1}$

Then, we can decompose the Between Treatment Sum of Squares into the Contrasts:

$SST = SSC_1 + \ldots + SSC_{t-1}$ where each of the Contrast Sums of Squares has 1 degree of freedom

For any contrast: Testing $H_{0k}: l_k = 0 \quad H_{Ak}: l_k \neq 0$

F-Test: $TS: F_k = \frac{SSC_k}{MSE} \quad RR: F_k \geq F_{\alpha,1,N-t}$ t-Test: $t_k = \frac{\hat{l}_k}{\hat{SE}\left\{\hat{l}_k\right\}} \quad RR: |t_k| \geq t_{\alpha/2;N-t}$

Simultaneous Tests of Multiple Contrasts

- Using m contrasts for comparisons among t treatments
- Each contrast to be tested at α significance level, which we label as α_I for individual comparison Type I error rate
- Probability of making at least one false rejection of one of the m null hypotheses is the experimentwise Type I error rate, which we label as α_E
- Tests are not independent unless the error (Within Group) degrees are infinite, however Bonferroni inequality implies that $\alpha_E \leq m\alpha_I \Rightarrow$ Choose $\alpha_I = \alpha_E / m$

Scheffe's Method for All Contrasts

- Can be used for any number of contrasts, even those suggested by data. Conservative (Wide CI's, Low Power)

$$l = \sum_{i=1}^{t} a_i \mu_i \quad s.t. \sum_{i=1}^{t} a_i = 0$$

$$\hat{l} = \sum_{i=1}^{t} a_i \overline{y}_{i\bullet} \quad \hat{SE}\left\{\hat{l}\right\} = \sqrt{MSE \sum_{i=1}^{t} \frac{a_i^2}{n_i}}$$

$$H_0 : l = \sum_{i=1}^{t} a_i \mu_i = 0 \quad H_A : l = \sum_{i=1}^{t} a_i \mu_i \neq 0$$

$$\text{Reject } H_0 \text{ if } \left|\hat{l}\right| \geq \hat{SE}\left\{\hat{l}\right\} \sqrt{(t-1) F_{\alpha_E, df_1, df_2}} \quad df_1 = t-1, \quad df_2 = df_{Error} = N - t$$

$$\text{Simultaneous } (1-\varepsilon)100\% \text{ Confidence Intervals: } \hat{l} \pm \hat{SE}\left\{\hat{l}\right\} \sqrt{(t-1) F_{\alpha_E, df_1, df_2}}$$

Post-hoc Comparisons of Treatments

- If differences in group means are determined from the *F*-test, researchers want to compare pairs of groups. Three popular methods include:
  - Fisher's LSD - Upon rejecting the null hypothesis of no differences in group means, LSD method is equivalent to doing pairwise comparisons among all pairs of groups as in Chapter 6.
  - Tukey's Method - Specifically compares all $t(t-1)/2$ pairs of groups. Utilizes a special table (Table 11, p. 701).
  - Bonferroni's Method - Adjusts individual comparison error rates so that all conclusions will be correct at desired confidence/significance level. Any number of comparisons can be made. Very general approach can be applied to any inferential problem

Fisher's Least Significant Difference Procedure

- Protected Version is to only apply method after significant result in overall *F*-test
- For each pair of groups, compute the **least significant difference (LSD)** that the sample means need to differ by to conclude the population means are not equal

$$LSD_{ij} = t_{\alpha/2}\sqrt{MSE\left(\frac{1}{n_i}+\frac{1}{n_j}\right)} \qquad \text{with df } = N - t$$

Conclude $\mu_i \neq \mu_j$ if $\left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq LSD_{ij}$

Tukey's *W* Procedure

- More conservative than Fisher's LSD (minimum significant difference and confidence interval width are higher).
- Derived so that the probability that at least one false difference is detected is α (experimentwise error rate)

$$W_{ij} = q_\alpha(t,\nu)\sqrt{\frac{MSE}{n}}$$

q given in Table 11, p. 701 with $\nu = N - t$

Conclude $\mu_i \neq \mu_j$ if $\left|\bar{y}_{i.} - \bar{y}_{j.}\right| \geq W_{ij}$

Tukey's Confidence Interval: $\left(\bar{y}_{i.} - \bar{y}_{j.}\right) \pm W_{ij}$

When the sample sizes are unequal, use $W_{ij} = \frac{q_\alpha(t,\nu)}{\sqrt{2}}\sqrt{MSE\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$

Bonferroni's Method (Most General)

- Wish to make C comparisons of pairs of groups with simultaneous confidence intervals or 2-sided tests

- When all pair of treatments are to be compared, $C = t(t-1)/2$

- Want the overall confidence level for all intervals to be "correct" to be 95% or the overall type I error rate for all tests to be 0.05

- For confidence intervals, construct (1-(0.05/C))100% CIs for the difference in each pair of group means (wider than 95% CIs)

- Conduct each test at α=0.05/C significance level (rejection region cut-offs more extreme than when α=0.05)

- Critical t-values are given in table on class website, we will use notation: $t_{\alpha/2,C,\nu}$ where C=#Comparisons, ν = df

Bonferroni's Method (Most General)

$$B_{ij} = t_{\alpha/2,C,v}\sqrt{MSE\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

(t given on class website with $v = N\text{-}t$)

Conclude $\mu_i \neq \mu_j$ if $\left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq B_{ij}$

Bonferroni's Confidence Interval : $\left(\overline{y}_{i.} - \overline{y}_{j.}\right) \pm B_{ij}$

Example - Seasonal Diet Patterns in Ravens

Note: No differences were found, these calculations are only for demonstration purposes

$$MSE = 0.03013 \quad n_i = 3 \quad t_{.025,8} = 2.306 \quad q_{.05,t=4,df_E=8} = 4.53 \quad t_{.025,C=6,df_E=8} = 3.479$$

$$LSD_{ij} = 2.306\sqrt{(0.03013)\left(\frac{1}{3}+\frac{1}{3}\right)} = 0.3268$$

$$W_{ij} = 4.53\sqrt{(0.03013)\left(\frac{1}{3}\right)} = 0.4540$$

$$B_{ij} = 3.479\sqrt{(0.03013)\left(\frac{1}{3}+\frac{1}{3}\right)} = 0.4930$$

Comparison(i vs j)	Group i Mean	Group j Mean	Difference
1 vs 2	1.243203	1.217935	0.025267
1 vs 3	1.243203	0.919375	0.323828
1 vs 4	1.243203	1.161773	0.081430
2 vs 3	1.217935	0.919375	0.298560
2 vs 4	1.217935	1.161773	0.056162
3 vs 4	0.919375	1.161773	-0.242398

Randomized Block Design (RBD)

- $t > 2$ Treatments (groups) to be compared
- b Blocks of homogeneous units are sampled. Blocks can be individual subjects. Blocks are made up of t subunits
- Subunits within a block receive one treatment. When subjects are blocks, receive treatments in random order.
- Outcome when Treatment i is assigned to Block j is labeled Y_{ij}
- Effect of Trt i is labeled α_i
- Effect of Block j is labeled β_j
- Random error term is labeled ε_{ij}
- Efficiency gain from removing block-to-block variability from experimental error

Randomized Complete Block Designs

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} = \mu_i + \beta_j + \varepsilon_{ij}$$

$$\sum_{i=1}^{t} \alpha_i = 0 \quad E(\varepsilon_{ij}) = 0 \qquad V(\varepsilon_{ij}) = \sigma^2$$

Note:

$$\overline{Y}_{1\bullet} = \frac{1}{b}\left[Y_{11} + \ldots + Y_{1b}\right] = \frac{1}{b}\left[\left(\mu + \alpha_1 + \beta_1 + \varepsilon_{11}\right) + \ldots + \left(\mu + \alpha_1 + \beta_b + \varepsilon_{1b}\right)\right] =$$

$$= \mu + \alpha_1 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{1\bullet}$$

$$\overline{Y}_{2\bullet} = \mu + \alpha_2 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{2\bullet}$$

$$\Rightarrow \quad \overline{Y}_{1\bullet} - \overline{Y}_{2\bullet} = \left(\mu + \alpha_1 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{1\bullet}\right) - \left(\mu + \alpha_2 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{2\bullet}\right) = \left(\alpha_1 - \alpha_2\right) + \left(\overline{\varepsilon}_{1\bullet} - \overline{\varepsilon}_{2\bullet}\right)$$

- Test for differences among treatment effects:
  - H_0: $\alpha_1 = \ldots = \alpha_t = 0 \quad (\mu_1 = \ldots = \mu_t)$
  - H_A: Not all $\alpha_i = 0$ (Not all μ_i are equal)

RBD - ANOVA *F*-Test (Normal Data)

- Data Structure: (*t* Treatments, *b* Blocks)
  - Mean for Treatment *i*: $\overline{y}_{i.}$
  - Mean for Subject (Block) *j*: $\overline{y}_{.j}$
  - Overall Mean: $\overline{y}_{..}$
  - Overall sample size: $N = bt$
- ANOVA:**T**reatment, **B**lock, and **E**rror Sums of Squares

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{b}\left(y_{ij} - \overline{y}_{..}\right)^2 \qquad df_{Total} = bt - 1$$

$$SST = b\sum_{i=1}^{t}\left(\overline{y}_{i.} - \overline{y}_{..}\right)^2 \qquad df_T = t - 1$$

$$SSB = t\sum_{j=1}^{b}\left(\overline{y}_{.j} - \overline{y}_{..}\right)^2 \qquad df_B = b - 1$$

$$SSE = \sum_{i=1}^{t}\sum_{j=1}^{b}\left(y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..}\right)^2 = TSS - SST - SSB \qquad df_E = (b-1)(t-1)$$

$$E\left\{MSE\right\} = \sigma^2 \qquad E\left\{MST\right\} = \sigma^2 + \frac{b\sum_{i=1}^{t}\alpha_i^2}{t-1}$$

RBD - ANOVA *F*-Test (Normal Data)

- ANOVA Table:

Source	SS	df	MS	*F*
Treatments	*SST*	*t*-1	*MST* = *SST*/(*t*-1)	*F* = *MST*/*MSE*
Blocks	*SSB*	*b*-1	*MSB* = *SSB*/(*b*-1)	
Error	*SSE*	(*b*-1)(*t*-1)	*MSE* = *SSE*/[(*b*-1)(*t*-1)]	
Total	*TSS*	*bt*-1		

- H_0: $\alpha_1 = \ldots = \alpha_t = 0$ $\quad(\mu_1 = \ldots = \mu_t)$
- H_A: Not all $\alpha_i = 0$ $\quad$(Not all μ_i are equal)

$$T.S.: F_{obs} = \frac{MST}{MSE}$$

$$R.R.: F_{obs} \geq F_{\alpha,t-1,(b-1)(t-1)}$$

$$P-val: P(F \geq F_{obs})$$

Pairwise Comparison of Treatment Means

- Tukey's Method- q in Studentized Range Table with $\nu = (b\text{-}1)(t\text{-}1)$

$$W_{ij} = q_{\alpha}(t,\nu)\sqrt{\frac{MSE}{b}}$$

$$\text{Conclude } \mu_i \neq \mu_j \text{ if } \left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq W_{ij}$$

$$\text{Tukey's Confidence Interval}: \left(\overline{y}_{i.} - \overline{y}_{j.}\right) \pm W_{ij}$$

- Bonferroni's Method - t-values from table on class website with $\nu = (b\text{-}1)(t\text{-}1)$ and $C=t(t\text{-}1)/2$

$$B_{ij} = t_{\alpha/2,C,\nu}\sqrt{\frac{2MSE}{b}}$$

$$\text{Conclude } \mu_i \neq \mu_j \text{ if } \left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq B_{ij}$$

$$\text{Bonferroni's Confidence Interval}: \left(\overline{y}_{i.} - \overline{y}_{j.}\right) \pm B_{ij}$$

Expected Mean Squares / Relative Efficiency

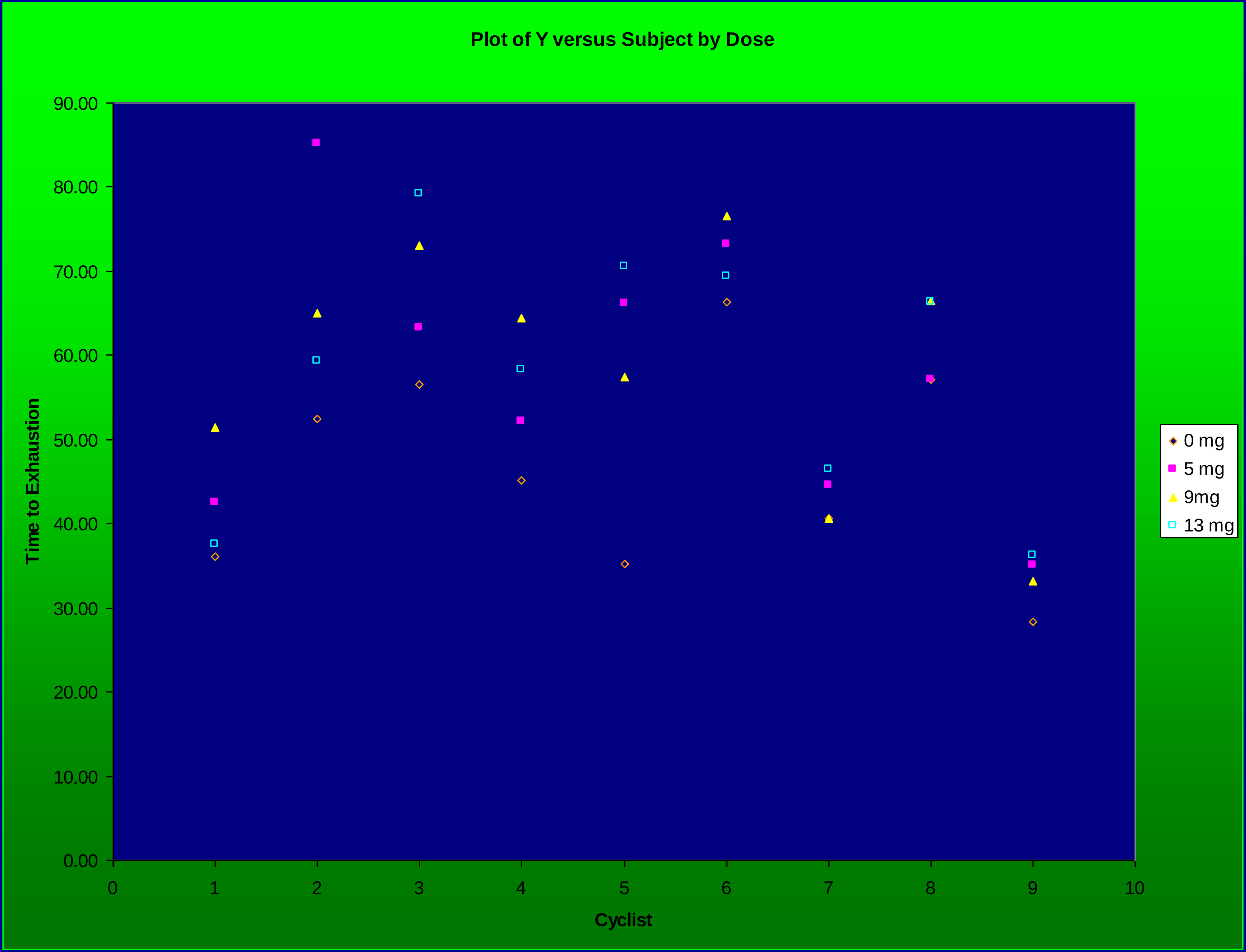
- Expected Mean Squares: As with CRD, the Expected Mean Squares for Treatment and Error are functions of the sample sizes (*b*, the number of blocks), the true treatment effects ($\alpha_1,\ldots,\alpha_t$) and the variance of the random error terms (σ^2)
- By assigning all treatments to units within blocks, error variance is (much) smaller for RBD than CRD (which combines block variation&random error into error term)
- Relative Efficiency of RBD to CRD (how many times as many replicates would be needed for CRD to have as precise of estimates of treatment means as RBD does):

$$RE(RCB,CR) = \frac{MSE_{CR}}{MSE_{RCB}} = \frac{(b-1)MSB + b(t-1)MSE}{(bt-1)MSE}$$

Example - Caffeine and Endurance

- Treatments: t=4 Doses of Caffeine: 0, 5, 9, 13 *mg*
- Blocks: b=9 Well-conditioned cyclists
- Response: y_{ij}=Minutes to exhaustion for cyclist j @ dose i
- Data:

Dose \ Subject	1	2	3	4	5	6	7	8	9
0	36.05	52.47	56.55	45.20	35.25	66.38	40.57	57.15	28.34
5	42.47	85.15	63.20	52.10	66.20	73.25	44.50	57.17	35.05
9	51.50	65.00	73.10	64.40	57.45	76.49	40.55	66.47	33.17
13	37.55	59.30	79.12	58.33	70.54	69.47	46.48	66.35	36.20



Plot of Y versus Subject by Dose
Time to Exhaustion
90.00
80.00
70.00
60.00
50.00
40.00
30.00
20.00
10.00
0.00
0
1
2
3
4
5
6
7
8
9
10
Cyclist
0 mg
5 mg
9mg
13 mg

Example - Caffeine and Endurance

Subject\Dose	0mg	5mg	9mg	13mg	Subj Mea	Subj Dev	Sqr Dev
1	36.05	42.47	51.50	37.55	41.89	-13.34	178.07
2	52.47	85.15	65.00	59.30	65.48	10.24	104.93
3	56.55	63.20	73.10	79.12	67.99	12.76	162.71
4	45.20	52.10	64.40	58.33	55.01	-0.23	0.05
5	35.25	66.20	57.45	70.54	57.36	2.12	4.51
6	66.38	73.25	76.49	69.47	71.40	16.16	261.17
7	40.57	44.50	40.55	46.48	43.03	-12.21	149.12
8	57.15	57.17	66.47	66.35	61.79	6.55	42.88
9	28.34	35.05	33.17	36.20	33.19	-22.05	486.06
Dose Mean	46.44	57.68	58.68	58.15	55.24		1389.50
Dose Dev	-8.80	2.44	3.44	2.91			
Squared Dev	77.38	5.95	11.86	8.48	103.68		
TSS	7752.773						

$$TSS = (36.05-55.24)^2 + \cdots + (36.20-55.24)^2 = 7752.773 \quad df_{Total} = 4(9)-1 = 35$$

$$SST = 9\left[(46.44-55.24)^2 + \cdots + (58.15-55.24)^2\right] = 9(103.68) = 933.12 \quad df_T = 4-1 = 3$$

$$SSB = 4\left[(41.89-55.24)^2 + \cdots + (33.19-55.24)^2\right] = 4(1389.50) = 5558.00 \quad df_B = 9-1 = 8$$

$$SSE = (36.05-41.89-46.44+55.24)^2 + \cdots + (36.20-33.19-58.15+55.24)^2 =$$

$$= TSS - SST - SSB = 7752.773 - 933.12 - 5558 = 1261.653 \quad df_E = (4-1)(9-1) = 24$$

Example - Caffeine and Endurance

Source	df	SS	MS	F
Dose	3	933.12	311.04	5.92
Cyclist	8	5558.00	694.75	
Error	24	1261.65	52.57	
Total	35	7752.77		

H_0 : No Caffeine Dose Effect $(\alpha_1 = \cdots = \alpha_4 = 0)$

H_A : Differences Exist Among Doses

$$T.S.: F_{obs} = \frac{MST}{MSE} = \frac{311.04}{52.57} = 5.92$$

$$R.R.(\alpha = 0.05): F_{obs} \geq F_{.05,3,24} = 3.01$$

$P-$value : $P(F \geq 5.92) = .0036$ (From EXCEL)

Conclude that true means are not all equal

Example - Caffeine and Endurance

$$\text{Tukey's } W: q_{.05,4,24} = 3.90 \quad W = 3.90\sqrt{52.57\left(\frac{1}{9}\right)} = 9.43$$

$$\text{Bonferroni's B}: t_{.05/2,6,24} = 2.875 \quad B = 2.875\sqrt{52.57\left(\frac{2}{9}\right)} = 9.83$$

Doses	High Mean	Low Mean	Difference	Conclude
5mg vs 0mg	57.6767	46.4400	11.2367	$\mu5>\mu0$
9mg vs 0mg	58.6811	46.4400	12.2411	$\mu9>\mu0$
13mg vs 0mg	58.1489	46.4400	11.7089	$\mu13>\mu0$
9mg vs 5mg	58.6811	57.6767	1.0044	NSD
13mg vs 5mg	58.1489	57.6767	0.4722	NSD
13mg vs 9mg	58.1489	58.6811	-0.5322	NSD

Example - Caffeine and Endurance

Relative Efficiency of Randomized Block to Completely Randomized Design :

$$t = 4 \quad b = 9 \quad MSB = 694.75 \quad MSE = 52.57$$

$$RE(RCB, CR) = \frac{(b-1)MSB + b(t-1)MSE}{(bt-1)MSE} = \frac{8(694.75) + 9(3)(52.57)}{(9(4)-1)(52.57)} = \frac{6977.39}{1839.95} = 3.79$$

Would have needed 3.79 times as many cyclists per dose to have the same precision on the estimates of mean endurance time.

- 9(3.79) ≈ 35 cyclists per dose
- 4(35) = 140 total cyclists

RBD -- Non-Normal Data
Friedman's Test

- When data are non-normal, test is based on ranks
- Procedure to obtain test statistic:
  - Rank the t treatments within each block (1=smallest, t=largest) adjusting for ties
  - Compute rank sums for treatments (T_i) across blocks
  - H_0: The t populations are identical ($M_1 = \ldots = M_t$)
  - H_A: Differences exist among the t group medians

$$T.S.: F_r = \frac{12}{bt(t+1)} \sum_{i=1}^{t} T_i^2 - 3b(t+1)$$

$$R.R.: F_r \geq \chi^2_{\alpha, t-1}$$

$$P-val : P(\chi^2 \geq F_r)$$

Example - Caffeine and Endurance

Subject\Dose	0mg	5mg	9mg	13mg	Ranks	0mg	5mg	9mg	13mg
1	36.05	42.47	51.5	37.55		1	3	4	2
2	52.47	85.15	65	59.3		1	4	3	2
3	56.55	63.2	73.1	79.12		1	2	3	4
4	45.2	52.1	64.4	58.33		1	2	4	3
5	35.25	66.2	57.45	70.54		1	3	2	4
6	66.38	73.25	76.49	69.47		1	3	4	2
7	40.57	44.5	40.55	46.48		2	3	1	4
8	57.15	57.17	66.47	66.35		1	2	4	3
9	28.34	35.05	33.17	36.2		1	3	2	4
					Total	10	25	27	28

H_0 : No Dose Differences

H_a : Dose Differences Exist

$$T.S.: F_r = \frac{12}{9(4)(4+1)}\left[(10)^2 + \cdots + (28)^2\right] - 3(9)(4+1) = \frac{26856}{180} - 135 = 14.2$$

$$R.R.(\alpha = 0.05): F_r \geq \chi^2_{.05,4-1} = 7.815$$

P-value: $P(\chi^2 \geq 14.2) = .0026$ (From EXCEL)

Conclude Medians are not all equal

Latin Square Design

- Design used to compare *t* treatments when there are two sources of extraneous variation (types of blocks), each observed at *t* levels
- Best suited for analyses when $t \leq 10$
- Classic Example: Car Tire Comparison
  - Treatments: 4 Brands of tires (A,B,C,D)
  - Extraneous Source 1: Car (1,2,3,4)
  - Extraneous Source 2: Position (Driver Front, Passenger Front, Driver Rear, Passenger Rear)

Car\Position	DF	PF	DR	PR
1	A	B	C	D
2	B	C	D	A
3	C	D	A	B
4	D	A	B	C

Latin Square Design - Model

- Model (t treatments, rows, columns, $N=t^2$) :

$$y_{ijk} = \mu + \alpha_k + \beta_i + \gamma_j + \varepsilon_{ijk}$$

$\mu \equiv$ Overall Mean $\quad \hat{\mu} = \overline{y}_{...}$

$\alpha_k \equiv$ Effect of Treatment k $\quad \hat{\alpha}_k = \overline{y}_{..k} - \overline{y}_{...}$

$\beta_i \equiv$ Effect due to row i $\quad \hat{\beta}_i = \overline{y}_{i..} - \overline{y}_{...}$

$\gamma_j \equiv$ Effect due to Column j $\quad \hat{\gamma}_j = \overline{y}_{.j.} - \overline{y}_{...}$

$\varepsilon_{ijk} \equiv$ Random Error Term

Latin Square Design - ANOVA & *F*-Test

Total Sum of Squares : $TSS = \sum_{i=1}^{t}\sum_{j=1}^{t}\left(y_{ijk} - \overline{y}_{...}\right)^2 \quad df = t^2 - 1$

Treatment Sum of Squares $SST = t\sum_{k=1}^{t}\left(\overline{y}_{..k} - \overline{y}_{...}\right)^2 \quad df_T = t - 1$

Row Sum of Squares $SSR = t\sum_{i=1}^{t}\left(\overline{y}_{i..} - \overline{y}_{...}\right)^2 \quad df_R = t - 1$

Column Sum of Squares $SSC = t\sum_{j=1}^{t}\left(\overline{y}_{.j.} - \overline{y}_{...}\right)^2 \quad df_C = t - 1$

Error Sum of Squares $SSE = TSS - SST - SSR - SSC \quad df_E = (t^2 - 1) - 3(t - 1) = (t - 1)(t - 2)$

- H_0: $\alpha_1 = \ldots = \alpha_t = 0$ H_a: Not all $\alpha_k = 0$
- TS: $F_{obs} = MST/MSE = (SST/(t-1))/(SSE/((t-1)(t-2)))$
- RR: $F_{obs} \geq F_{\alpha, t-1, (t-1)(t-2)}$

Pairwise Comparison of Treatment Means

- Tukey's Method- q in Studentized Range Table with $\nu = (t\text{-}1)(t\text{-}2)$

$$W_{ij} = q_{\alpha}(t,v)\sqrt{\frac{MSE}{t}}$$

$$\text{Conclude } \mu_i \neq \mu_j \text{ if } \left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq W_{ij}$$

$$\text{Tukey's Confidence Interval}: \left(\overline{y}_{i.} - \overline{y}_{j.}\right) \pm W_{ij}$$

- Bonferroni's Method - t-values from table on class website with $\nu = (t\text{-}1)(t\text{-}2)$ and $C=t(t\text{-}1)/2$

$$B_{ij} = t_{\alpha/2,C,v}\sqrt{\frac{2MSE}{t}}$$

$$\text{Conclude } \mu_i \neq \mu_j \text{ if } \left|\overline{y}_{i.} - \overline{y}_{j.}\right| \geq B_{ij}$$

$$\text{Bonferroni's Confidence Interval}: \left(\overline{y}_{i.} - \overline{y}_{j.}\right) \pm B_{ij}$$

Expected Mean Squares / Relative Efficiency

- Expected Mean Squares: As with CRD, the Expected Mean Squares for Treatment and Error are functions of the sample sizes (t, the number of blocks), the true treatment effects ($\alpha_1,\ldots,\alpha_t$) and the variance of the random error terms (σ^2)
- By assigning all treatments to units within blocks, error variance is (much) smaller for LS than CRD (which combines block variation&random error into error term)
- Relative Efficiency of LS to CRD (how many times as many replicates would be needed for CRD to have as precise of estimates of treatment means as LS does):

$$RE(LS,CR) = \frac{MSE_{CR}}{MSE_{LS}} = \frac{MSR + MSC + (t-1)MSE}{(t+1)MSE}$$

Power Approach to Sample Size Choice – R Code

When the means are not all equal, the F-statistic is non-central F:

$$F^* \sim F\left(t-1, N-t, \lambda\right) \quad \text{where } \lambda = \frac{\sum_{i=1}^{t} n_i \left(\mu_i - \mu_\bullet\right)^2}{\sigma^2} \quad \text{where } \mu_\bullet = \frac{\sum_{i=1}^{t} n_i \mu_i}{N}$$

When all sample sizes are equal: $\lambda = \dfrac{n\sum_{i=1}^{t}\left(\mu_i - \mu_\bullet\right)^2}{\sigma^2}$ where $\mu_\bullet = \dfrac{\sum_{i=1}^{t} \mu_i}{t}$

The power of the test, when conducted at the significance level of α:

$$\Pr\left\{F^* \geq F\left(1-\alpha; t-1, N-t\right) \mid F^* \sim F\left(t-1, N-t, \lambda\right)\right\}$$

In R:

$$F\left(1-\alpha; t-1, N-t\right) = qf\,(1-\alpha, t-1, N-t)$$

$$\text{Power} = 1-\beta = 1 - pf\left(qf\,(1-\alpha, t-1, N-t), t-1, N-t, \lambda\right)$$

2-Way ANOVA

- 2 nominal or ordinal factors are believed to be related to a quantitative response
- Additive Effects: The effects of the levels of each factor do not depend on the levels of the other factor.
- Interaction: The effects of levels of each factor depend on the levels of the other factor
- Notation: μ_{ij} is the mean response when factor A is at level *i* and Factor B at *j*

2-Way ANOVA - Model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,a \quad j = 1,\ldots,b \quad k = 1,\ldots,n$$

$y_{ijk} \equiv$ Measurement on k^{th} unit receiving Factors A at level i, B at level j

$\mu \equiv$ Overall Mean

$\alpha_i \equiv$ Effect of i^{th} level of factor A

$\beta_j \equiv$ Effect of j^{th} level of factor B

$\alpha\beta_{ij} \equiv$ Interaction effect when i^{th} level of A and j^{th} level of B are combined

$\varepsilon_{ijk} \equiv$ Random Error Terms

- Model depends on whether all levels of interest for a factor are included in experiment:
  - **Fixed Effects:** All levels of factors A and B included
  - **Random Effects:** Subset of levels included for factors A and B
  - **Mixed Effects:** One factor has all levels, other factor a subset

Fixed Effects Model

- Factor A: Effects are fixed constants and sum to 0
- Factor B: Effects are fixed constants and sum to 0
- Interaction: Effects are fixed constants and sum to 0 over all levels of factor B, for each level of factor A, and vice versa
- Error Terms: Random Variables that are assumed to be independent and normally distributed with mean 0, variance σ_ε^2

$$\sum_{i=1}^{a}\alpha_i = 0, \quad \sum_{j=1}^{b}\beta_j = 0 \quad \sum_{i=1}^{a}\alpha\beta_{ij} = 0\ \forall j \quad \sum_{j=1}^{b}\alpha\beta_{ij} = 0\ \forall i \quad \varepsilon_{ijk} \sim N\left(0,\sigma_\varepsilon^2\right)$$

Example - Thalidomide for AIDS

- Response: 28-day weight gain in AIDS patients
- Factor A: Drug: Thalidomide/Placebo
- Factor B: TB Status of Patient: TB^+/TB^-
- Subjects: 32 patients (16 TB^+ and 16 TB^-). Random assignment of 8 from each group to each drug). Data:
  - Thalidomide/TB^+: 9,6,4.5,2,2.5,3,1,1.5
  - Thalidomide/TB^-: 2.5,3.5,4,1,0.5,4,1.5,2
  - Placebo/TB^+: 0,1,-1,-2,-3,-3,0.5,-2.5
  - Placebo/TB^-: -0.5,0,2.5,0.5,-1.5,0,1,3.5

ANOVA Approach

- Total Variation (*TSS*) is partitioned into 4 components:
  - Factor A: Variation in means among levels of A
  - Factor B: Variation in means among levels of B
  - Interaction: Variation in means among combinations of levels of A and B that are not due to A or B alone
  - Error: Variation among subjects within the same combinations of levels of A and B (Within SS)

Analysis of Variance

Total Variation: $TSS = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{...}\right)^2 \quad df_{Total} = abn - 1$

Factor A Sum of Squares: $SSA = bn\sum_{i=1}^{a}\left(\bar{y}_{i..} - \bar{y}_{...}\right)^2 \quad df_A = a - 1$

Factor B Sum of Squares: $SSB = an\sum_{j=1}^{b}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2 \quad df_B = b - 1$

Interaction Sum of Squares: $SSAB = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2 \quad df_{AB} = (a-1)(b-1)$

Error Sum of Squares: $SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{ij.}\right)^2 \quad df_E = ab(n-1)$

- $TSS = SSA + SSB + SSAB + SSE$
- $df_{Total} = df_A + df_B + df_{AB} + df_E$

ANOVA Approach - Fixed Effects

Source	df	SS	MS	F
Factor A	a-1	SSA	MSA=SSA/(a-1)	F_A=MSA/MSE
Factor B	b-1	SSB	MSB=SSB/(b-1)	F_B=MSB/MSE
Interaction	(a-1)(b-1)	SSAB	MSAB=SSAB/[(a-1)(b-1)]	F_{AB}=MSAB/MSE
Error	ab(n-1)	SSE	MSE=SSE/[ab(n-1)]	
Total	abn-1	TSS		

- Procedure:
  - First test for interaction effects
  - If interaction test not significant, test for Factor A and B effects

Test for Interaction:

$$H_0: \alpha\beta_{11} = \ldots = \alpha\beta_{ab} = 0$$

$$H_a: \text{ Not all } \alpha\beta_{ij} = 0$$

$$TS: F_{AB} = \frac{MSAB}{MSE}$$

$$RR: F_{AB} \geq F_{\alpha,(a-1)(b-1),ab(n-1)}$$

Test for Factor A

$$H_0: \alpha_1 = \ldots = \alpha_a = 0$$

$$H_a: \text{ Not all } \alpha_i = 0$$

$$TS: F_A = \frac{MSA}{MSE}$$

$$RR: F_A \geq F_{\alpha,(a-1),ab(n-1)}$$

Test for Factor B

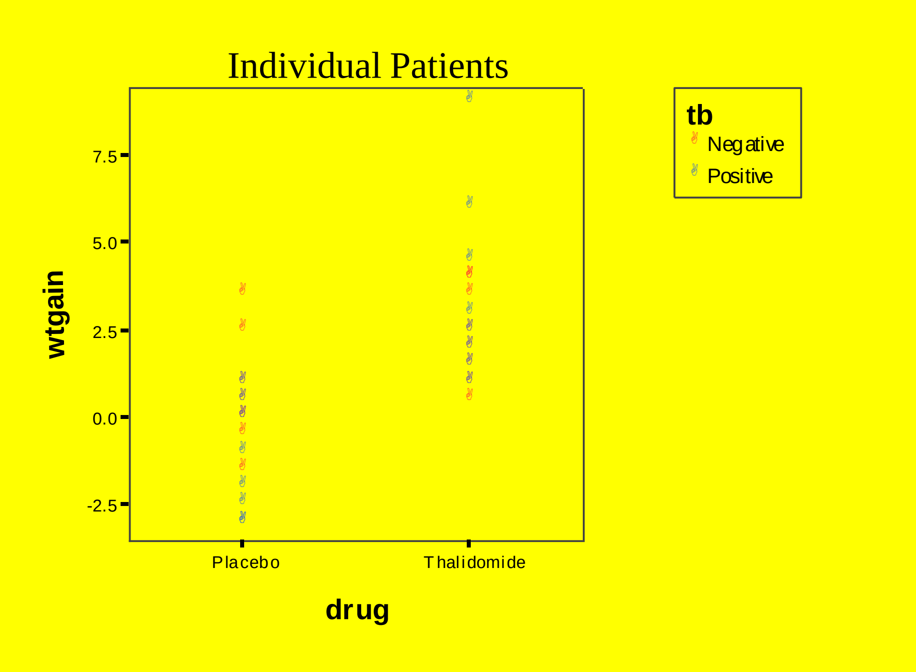
$$H_0: \beta_1 = \ldots = \beta_b = 0$$

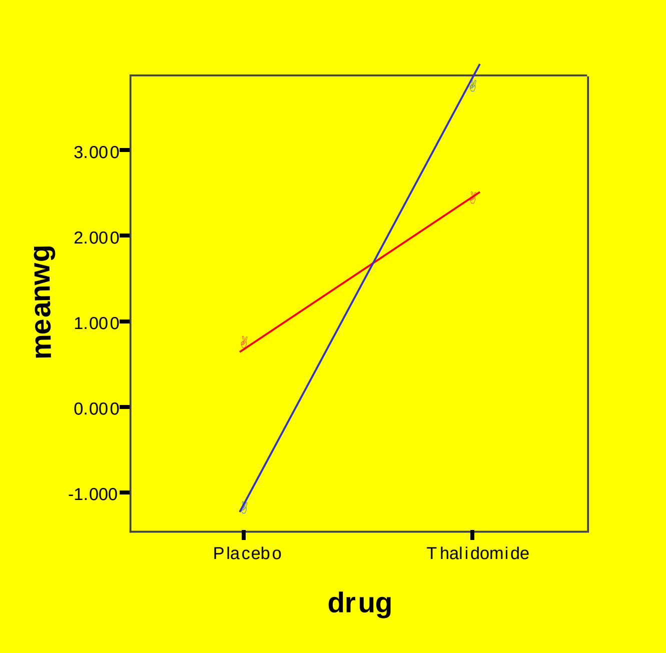
$$H_a: \text{ Not all } \beta_j = 0$$

$$TS: F_B = \frac{MSB}{MSE}$$

$$RR: F_B \geq F_{\alpha(b-1),ab(n-1)}$$

Example - Thalidomide for AIDS





Report

WTGAIN

GROUP	Mean	N	Std. Deviation
TB+/Thalidomide	3.688	8	2.6984
TB-/Thalidomide	2.375	8	1.3562
TB+/Placebo	-1.250	8	1.6036
TB-/Placebo	.688	8	1.6243
Total	1.375	32	2.6027

Example - Thalidomide for AIDS

Tests of Between-Subjects Effects

Dependent Variable: WTGAIN

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	109.688[a]	3	36.563	10.206	.000
Intercept	60.500	1	60.500	16.887	.000
DRUG	87.781	1	87.781	24.502	.000
TB	.781	1	.781	.218	.644
DRUG * TB	21.125	1	21.125	5.897	.022
Error	100.313	28	3.583		
Total	270.500	32			
Corrected Total	210.000	31			

a. R Squared = .522 (Adjusted R Squared = .471)

- There is a significant Drug*TB interaction (F_{DT}=5.897, P=.022)
- The Drug effect depends on TB status (and vice versa)

ANOVA – Additive Model

If the Interaction is not significant, the Interaction term can be removed, and an additive model can be fit.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \qquad SSE_A = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{i\bullet\bullet} - \overline{Y}_{\bullet j\bullet} + \overline{Y}_{\bullet\bullet\bullet}\right)^2 \qquad df_{E_A} = abn - a - b + 1$$

Source	df	SS	MS	F
Factor A	a-1	SSA	MSA=SSA/(a-1)	F_A=MSA/MSE_A
Factor B	b-1	SSB	MSB=SSB/(b-1)	F_B=MSB/MSE_A
Error	abn-a-b+1	SSE_A	MSE_A=SSE_A/[abn-a-b+1]	
Total	abn-1	TSS		

Comparing Main Effects (No Interaction)

- Tukey's Method- q in Studentized Range Table with $\nu = abn\text{-}a\text{-}b+1$

$$W_{ij}^{A} = q_{\alpha}(a,\nu)\sqrt{\frac{MSE_A}{bn}} \quad W_{ij}^{B} = q_{\alpha}(b,\nu)\sqrt{\frac{MSE_A}{an}}$$

Conclude: $\alpha_i \neq \alpha_j$ if $\left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq W_{ij}^{A}$ $\quad \beta_i \neq \beta_j$ if $\left|\overline{y}_{.i.} - \overline{y}_{.j.}\right| \geq W_{ij}^{B}$

Tukey's CI: $(\alpha_i - \alpha_j) : \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm W_{ij}^{A}$ $\quad (\beta_i - \beta_j) : \left(\overline{y}_{.i.} - \overline{y}_{.j.}\right) \pm W_{ij}^{B}$

Bonferroni's Method - t-values in Bonferroni table with $\nu = abn\text{-}a\text{-}b+1$

$$B_{ij}^{A} = t_{\alpha/2, a(a-1)/2, \nu}\sqrt{\frac{2MSE_A}{bn}} \quad B_{ij}^{B} = t_{\alpha/2, b(b-1)/2, \nu}\sqrt{\frac{2MSE_A}{an}}$$

Conclude: $\alpha_i \neq \alpha_j$ if $\left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq B_{ij}^{A}$ $\quad \beta_i \neq \beta_j$ if $\left|\overline{y}_{.i.} - \overline{y}_{.j.}\right| \geq B_{ij}^{B}$

Bonferroni's CI: $(\alpha_i - \alpha_j)$: $\left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm B_{ij}^{A}$ $\quad (\beta_i - \beta_j)$: $\left(\overline{y}_{.i.} - \overline{y}_{.j.}\right) \pm B_{ij}^{B}$

Comparing Main Effects (Interaction)

- Tukey's Method- q in Studentized Range Table with $\nu = ab(n\text{-}1)$

$$W_{ij}^{A} = q_{\alpha}(a,\nu)\sqrt{\frac{MSE}{n}}$$

Within kth level of Factor B, Conclude: $\alpha_i \neq \alpha_j$ if $\left|\overline{y}_{ik.} - \overline{y}_{jk.}\right| \geq W_{ij}^{A}$

Tukey's CI: $(\alpha_i - \alpha_j): \left(\overline{y}_{ik.} - \overline{y}_{jk.}\right) \pm W_{ij}^{A}$ Similar for Factor B in A

- Bonferroni's Method - t-values in Bonferroni table with $\nu = ab\,(n\text{-}1)$

$$B_{ij}^{A} = t_{\alpha/2,a(a-1)/2,\nu}\sqrt{\frac{2MSE}{n}}$$

Within kth level of B, Conclude: $\alpha_i \neq \alpha_j$ if $\left|\overline{y}_{ik.} - \overline{y}_{jk.}\right| \geq B_{ij}^{A}$

Bonferroni's CI:$(\alpha_i - \alpha_j)$: $\left(\overline{y}_{ik.} - \overline{y}_{jk.}\right) \pm B_{ij}^{A}$

Miscellaneous Topics

- 2-Factor ANOVA can be conducted in a Randomized Block Design, where each block is made up of *ab* experimental units. Analysis is direct extension of RBD with 1-factor ANOVA
- Factorial Experiments can be conducted with any number of factors. Higher order interactions can be formed (for instance, the *AB* interaction effects may differ for various levels of factor *C*).
- When experiments are not balanced, calculations are immensely messier and you must use statistical software packages for calculations

Unequal Sample Sizes

- When sample sizes are unequal, calculations and parameter interpretations (especially marginal ones) become messier
- Observational studies often have unequal sample sizes due to availability of sampling units for certain combinations of factor levels (villagers of certain types in a rural study for instance)
- Experimental studies, even when planned with equal sample sizes can end up unbalanced through technical problems or “drop outs”
- Some conditions may be cheaper to measure than others, and will have larger sample sizes
- Some situations have particular contrasts of higher importance

Regression Approach - I

Sample Sizes: # of Cases when Factor *A* is at level *i*, *B* @ *j*: n_{ij}

$$n_{i\bullet} = \sum_{j=1}^{b} n_{ij} \qquad n_{\bullet j} = \sum_{i=1}^{a} n_{ij} \qquad n_T = \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij} \qquad Y_{ij\bullet} = \sum_{k=1}^{n_{ij}} Y_{ijk} \qquad \overline{Y}_{ij\bullet} = \frac{Y_{ij\bullet}}{n_{ij}}$$

Model: $Y_{ijk} = \mu_{\bullet\bullet} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \qquad \varepsilon_{ijk} \sim N\left(0, \sigma^2\right)$ (independent)

Restrictions on Effects: $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} (\alpha\beta)_{ij} = \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0$

$$\Rightarrow \quad \alpha_a = -\alpha_1 - \alpha_2 - \ldots - \alpha_{a-1}$$

$$\Rightarrow \quad \beta_b = -\beta_1 - \beta_2 - \ldots - \beta_{b-1}$$

$$\Rightarrow \quad (\alpha\beta)_{ib} = -(\alpha\beta)_{i1} - (\alpha\beta)_{i2} - \ldots - (\alpha\beta)_{i,b-1}$$

$$\Rightarrow \quad (\alpha\beta)_{aj} = -(\alpha\beta)_{1j} - (\alpha\beta)_{2j} - \ldots - (\alpha\beta)_{a-1\,j}$$

Regression Approach - II

Regression Model:

$$Y_{ijk} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + \ldots + \alpha_{a-1} X_{ijk,a-1} + \beta_1 X_{ijka} + \ldots + \beta_{b-1} X_{ijk,a+b-2} +$$
$$(\alpha\beta)_{11} X_{ijk1} X_{ijka} + \ldots + (\alpha\beta)_{a-1,b-1} X_{ijk,a-1} X_{ijk,a+b-2} + \varepsilon_{ijk}$$

where: $$X_{ijk1} = \begin{cases} 1 & \text{if case from level 1 of factor A} \\ -1 & \text{if case from level a of factor A} \\ 0 & \text{otherwise} \end{cases}$$

$$X_{ijk,a-1} = \begin{cases} 1 & \text{if case from level a-1 of factor A} \\ -1 & \text{if case from level a of factor A} \\ 0 & \text{otherwise} \end{cases}$$

where: $$X_{ijka} = \begin{cases} 1 & \text{if case from level 1 of factor B} \\ -1 & \text{if case from level b of factor B} \\ 0 & \text{otherwise} \end{cases}$$

$$X_{ijk,a+b-2} = \begin{cases} 1 & \text{if case from level b-1 of factor B} \\ -1 & \text{if case from level b of factor B} \\ 0 & \text{otherwise} \end{cases}$$

Regression Approach – Example I

	Writer Type (B)			
Style (Factor A)	Poets (B1)	Year at Peak	Novelists (B2)	Year at Peak
Conceptualists (A1)	Eliot	23	Fitzgerald	29
(Finders)	Cummings	26	Hemingway	30
	Plath	30	Melville	32
	Pound	30	Lawrence	35
	Wilbur	34	Joyce	40
n & Mean	n11=5	28.60	n12=5	33.20
Experimentalists (A2)	Bishop	29	James	38
(Seekers)	Moore	32	Faulkner	39
	Williams	40	Dickens	41
	Lowell	41	Woolf	45
	Stevens	42	Conrad	47
	Frost	48	Twain	50
			Hardy	51
n & Mean	n21=6	38.67	n22=7	44.43

X_ijk1	X_ijk2	X1X2
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
1	-1	-1
1	-1	-1
1	-1	-1
1	-1	-1
1	-1	-1
-1	1	-1
-1	1	-1
-1	1	-1
-1	1	-1
-1	1	-1
-1	1	-1
-1	-1	1
-1	-1	1
-1	-1	1
-1	-1	1
-1	-1	1
-1	-1	1
-1	-1	1

$$Y_{ijk} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + \varepsilon_{ijk}$$

$$\alpha_2 = -\alpha_1 \qquad \beta_2 = -\beta_1$$

$$(\alpha\beta)_{12} = (\alpha\beta)_{21} = -(\alpha\beta)_{11} \qquad (\alpha\beta)_{22} = (\alpha\beta)_{11}$$

Testing Strategies – Models Fit

Model 1: all Factor A, Factor B, and Interaction AB Effects

Model 2:all Factor A, Factor B Effects (Remove Interaction)

Model 3: all Factor B,Interaction AB Effects (Remove A)

Model 4:all Factor A,Interaction AB Effects (Remove B)

To test for Interaction Effects, Model 1 is Full Model, Model 2 is Reduced $df_{Numerator}=(a-1)(b-1)$ $df_{den}=n_T-ab$

Testing for Factor A Effects, Full=Model 1, Reduced=Model 3
$df_{Numerator}=(a-1)$ $df_{den}=n_T-ab$

Testing for Factor B Effects, Full=Model 1, Reduced=Model 4
$df_{Numerator}=(b-1)$ $df_{den}=n_T-ab$

Regression Approach – Example - Continued

$$Y_{ijk} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + \varepsilon_{ijk}$$

Model 1: $E\{Y_{ijk}\} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2}$

$$\hat{Y} = 36.22 - 5.32X_1 - 2.59X_2 + 0.29X_1X_2$$

Model 2: $E\{Y_{ijk}\} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2}$ $\quad \hat{Y} = 36.23 - 5.33X_1 - 2.63X_2$

Model 3: $E\{Y_{ijk}\} = \mu_{\bullet\bullet} + \beta_1 X_{ijk2} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2}$ $\quad \hat{Y} = 36.90 - 2.77X_2 + 0.47X_1X_2$

Model 4: $E\{Y_{ijk}\} = \mu_{\bullet\bullet} + \alpha_1 X_{ijk1} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2}$ $\quad \hat{Y} = 36.31 - 5.41X_1 + 0.62X_1X_2$

ANOVA	Model1			Model2			Model3			Model4	
	df	*SS*		*df*	*SS*		*df*	*SS*		*df*	*SS*
Regression	3	827.91		2	826.01		2	188.77		2	676.58
Residual	19	557.05		20	558.95		20	1196.19		20	708.37
Total	22	1384.96		22	1384.96		22	1384.96		22	1384.96

Regression Approach – Example - Continued

$$H_0: (\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{21} = (\alpha\beta)_{22} = 0 \qquad H_A: \text{ Interaction Exists}$$

$$SSE(R) = 558.95 \quad df_E(R) = 20 \qquad SSE(F) = 557.05 \quad df_E(F) = 19$$

$$TS: F_{AB}^* = \frac{\left[\left(SSE(R) - SSE(F)\right) / \left(df_E(R) - df_E(F)\right)\right]}{\left[SSE(F)/df_E(F)\right]} =$$

$$\frac{\left[(558.95 - 557.05)/(20-19)\right]}{\left[557.05/19\right]} = 0.065 \quad RR: F_{AB}^* \geq F(.95,1,19) = 4.381$$

ANOVA	Model1		Model2		Model3		Model4	
	df	*SS*	*df*	*SS*	*df*	*SS*	*df*	*SS*
Regression	3	827.91	2	826.01	2	188.77	2	676.58
Residual	19	557.05	20	558.95	20	1196.19	20	708.37
Total	22	1384.96	22	1384.96	22	1384.96	22	1384.96

Regression Approach – Example - Continued

$H_0: \alpha_1 = \alpha_2 = 0 \quad H_A:$ Factor A Effects Exist:

$SSE(R) = 1196.19 \quad df_E(R) = 20$

$$F_A^* = \frac{\left[(1196.19 - 557.05)/(20-19)\right]}{[557.05/19]} = 21.80 \quad RR: F_A^* \geq F(.95,1,19) = 4.381$$

$H_0: \beta_1 = \beta_2 = 0 \quad H_A:$ Factor B Effects Exist:

$SSE(R) = 708.37 \quad df_E(R) = 20$

$$F_B^* = \frac{\left[(708.37 - 557.05)/(20-19)\right]}{[557.05/19]} = 5.16 \quad RR: F_B^* \geq F(.95,1,19) = 4.381$$

ANOVA	Model1			Model2			Model3			Model4	
	df	*SS*		*df*	*SS*		*df*	*SS*		*df*	*SS*
Regression	3	827.91		2	826.01		2	188.77		2	676.58
Residual	19	557.05		20	558.95		20	1196.19		20	708.37
Total	22	1384.96		22	1384.96		22	1384.96		22	1384.96

Estimating Treatment and Factor Level Means/Contrasts

Treatment Means:

Parameter: μ_{ij} Estimator: $\hat{\mu}_{ij} = \frac{\sum_{k=1}^{n_{ij}} Y_{ijk}}{n_{ij}} = \overline{Y}_{ij\bullet}$ Estimated Standard Error: $s\left\{\hat{\mu}_{ij}\right\} = \sqrt{\frac{MSE}{n_{ij}}}$

Factor A Means:

Parameter: $\mu_{i\bullet} = \frac{\sum_{j=1}^{b} \mu_{ij}}{b}$ Estimator: $\hat{\mu}_{i\bullet} = \frac{\sum_{j=1}^{b} \overline{Y}_{ij\bullet}}{b}$ Estimated Standard Error: $s\left\{\hat{\mu}_{i\bullet}\right\} = \sqrt{\frac{MSE}{b^2} \sum_{j=1}^{b} \frac{1}{n_{ij}}}$

Factor B Means:

Parameter: $\mu_{\bullet j} = \frac{\sum_{i=1}^{a} \mu_{ij}}{a}$ Estimator: $\hat{\mu}_{\bullet j} = \frac{\sum_{i=1}^{a} \overline{Y}_{ij\bullet}}{a}$ Estimated Standard Error: $s\left\{\hat{\mu}_{\bullet j}\right\} = \sqrt{\frac{MSE}{a^2} \sum_{i=1}^{a} \frac{1}{n_{ij}}}$

Contrast or Linear Function of Factor A Means:

Parameter: $L_A = \sum_{i=1}^{a} c_i \mu_{i\bullet}$ Estimator: $\hat{L}_A = \sum_{i=1}^{a} c_i \hat{\mu}_{i\bullet}$ Estimated Standard Error: $s\left\{\hat{L}_A\right\} = \sqrt{\frac{MSE}{b^2} \sum_{i=1}^{a} c_i^2 \sum_{j=1}^{b} \frac{1}{n_{ij}}}$

Contrast or Linear Function of Factor B Means:

Parameter: $L_B = \sum_{j=1}^{b} c_j \mu_{\bullet j}$ Estimator: $\hat{L}_B = \sum_{j=1}^{b} c_j \hat{\mu}_{\bullet j}$ Estimated Standard Error: $s\left\{\hat{L}_B\right\} = \sqrt{\frac{MSE}{a^2} \sum_{j=1}^{b} c_j^2 \sum_{i=1}^{a} \frac{1}{n_{ij}}}$

Contrast or Linear Function of Treatment Means:

Parameter: $L_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} c_{ij} \mu_{ij}$ Estimator: $\hat{L}_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} c_{ij} \overline{Y}_{ij\bullet}$ Estimated Standard Error: $s\left\{\hat{L}_{AB}\right\} = \sqrt{MSE \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{c_{ij}^2}{n_{ij}}}$

Standard Error Multipliers

Single Comparisons:

$t\left(\alpha/2; N-ab\right)$

General Multiple Comparisons of Treatment (Cell) Means :

Scheffe: $S = \sqrt{(ab-1)F\left(\alpha; ab-1, N-ab\right)}$

Bonferroni: $B = t\left(\alpha/(2g), N-ab\right)$ $\quad g =$ # of comparisons

Tukey (all pairs of treatment means): $T = \frac{1}{\sqrt{2}} q\left(\alpha; ab, N-ab\right)$

General Multiple Comparisons of Factor Level Means :

Scheffe: Factor A: $S_A = \sqrt{(a-1)F\left(\alpha; a-1, N-ab\right)}$ Factor B: $S_B = \sqrt{(b-1)F\left(1-\alpha; b-1, N-ab\right)}$

Bonferroni: Factor A or Factor B : $B = t\left(\alpha/(2g), N-ab\right)$ $\quad g =$ # of comparisons

Tukey: Factor A: $T_A = \frac{1}{\sqrt{2}} q\left(\alpha; a, N-ab\right)$ Factor B: $T_B = \frac{1}{\sqrt{2}} q\left(\alpha; b, N-ab\right)$

Creative Life Cycles – Comparing Treatment Means

Comparing all 4 Treatment Means(athough no interaction was present):

$\overline{Y}_{11\bullet} = 28.60 \quad n_{11} = 5 \quad s\left\{\overline{Y}_{11\bullet}\right\} = \sqrt{\frac{MSE}{n_{11}}} = \sqrt{\frac{29.32}{5}} = 2.42 \qquad \overline{Y}_{12\bullet} = 33.20 \quad n_{12} = 5 \quad s\left\{\overline{Y}_{12\bullet}\right\} = \sqrt{\frac{MSE}{n_{12}}} = \sqrt{\frac{29.32}{5}} = 2.42$

$\overline{Y}_{21\bullet} = 38.67 \quad n_{21} = 6 \quad s\left\{\overline{Y}_{21\bullet}\right\} = \sqrt{\frac{MSE}{n_{21}}} = \sqrt{\frac{29.32}{6}} = 2.21 \qquad \overline{Y}_{22\bullet} = 44.43 \quad n_{22} = 7 \quad s\left\{\overline{Y}_{22\bullet}\right\} = \sqrt{\frac{MSE}{n_{22}}} = \sqrt{\frac{29.32}{7}} = 2.05$

$T = \frac{1}{\sqrt{2}} q\left(0.95, 4, 23-4=19\right) = \frac{1}{\sqrt{2}}\left(3.977\right) = 2.812 \quad s\left\{\overline{Y}_{ij\bullet} - \overline{Y}_{i'j'\bullet}\right\} = \sqrt{MSE\left(\frac{1}{n_{ij}} + \frac{1}{n_{i'j'}}\right)}$

$\overline{Y}_{11\bullet} - \overline{Y}_{12\bullet} = 28.60 - 33.20 = -4.60 \quad s\left\{\overline{Y}_{11\bullet} - \overline{Y}_{12\bullet}\right\} = \sqrt{29.32\left(\frac{1}{5} + \frac{1}{5}\right)} = 3.43 \qquad HSD = 2.812(3.43) = 9.65$

$\overline{Y}_{11\bullet} - \overline{Y}_{21\bullet} = 28.60 - 38.67 = -10.07 \quad s\left\{\overline{Y}_{11\bullet} - \overline{Y}_{21\bullet}\right\} = \sqrt{29.32\left(\frac{1}{5} + \frac{1}{6}\right)} = 3.28 \qquad HSD = 2.812(3.28) = 9.22$

$\overline{Y}_{11\bullet} - \overline{Y}_{22\bullet} = 28.60 - 44.43 = -15.83 \quad s\left\{\overline{Y}_{11\bullet} - \overline{Y}_{22\bullet}\right\} = \sqrt{29.32\left(\frac{1}{5} + \frac{1}{7}\right)} = 3.17 \qquad HSD = 2.812(3.17) = 8.92$

$\overline{Y}_{12\bullet} - \overline{Y}_{21\bullet} = 33.20 - 38.67 = -5.47 \quad s\left\{\overline{Y}_{12\bullet} - \overline{Y}_{21\bullet}\right\} = \sqrt{29.32\left(\frac{1}{5} + \frac{1}{6}\right)} = 3.28 \qquad HSD = 2.812(3.28) = 9.22$

$\overline{Y}_{12\bullet} - \overline{Y}_{22\bullet} = 33.20 - 44.43 = -11.23 \quad s\left\{\overline{Y}_{12\bullet} - \overline{Y}_{22\bullet}\right\} = \sqrt{29.32\left(\frac{1}{5} + \frac{1}{7}\right)} = 3.17 \qquad HSD = 2.812(3.17) = 8.92$

$\overline{Y}_{21\bullet} - \overline{Y}_{22\bullet} = 38.67 - 44.43 = -5.76 \quad s\left\{\overline{Y}_{21\bullet} - \overline{Y}_{22\bullet}\right\} = \sqrt{29.32\left(\frac{1}{6} + \frac{1}{7}\right)} = 3.01 \qquad HSD = 2.812(3.01) = 8.47$

Conceptualists/Poets **Conceptualists/Novelists** **Experimentalists/Poets** **Experimentalists/Novelists**

Creative Life Cycles – Comparing Factor Level Means

Factor A (Style):

$$\hat{\mu}_{1\bullet} = \frac{\sum_{j=1}^{b} \overline{Y}_{1j\bullet}}{b} = \frac{\overline{Y}_{11\bullet} + \overline{Y}_{12\bullet}}{2} = \frac{28.60 + 33.20}{2} = 30.90$$

$$\hat{\mu}_{2\bullet} = \frac{\sum_{j=1}^{b} \overline{Y}_{2j\bullet}}{b} = \frac{\overline{Y}_{21\bullet} + \overline{Y}_{22\bullet}}{2} = \frac{38.67 + 44.43}{2} = 41.55$$

$$\hat{\mu}_{1\bullet} - \hat{\mu}_{2\bullet} = 30.9 - 41.55 = -10.65$$

$$s\left\{\hat{\mu}_{1\bullet} - \hat{\mu}_{2\bullet}\right\} = \sqrt{\frac{29.32}{2^2}\left(1^2 + (-1)^2\right)\left(\frac{1}{5} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right)} = \sqrt{10.402} = 3.23$$

$$t\left(0.975, 23 - 4 = 19\right) = 2.093$$

95% CI for $\mu_{1\bullet} - \mu_{2\bullet}$ (Conceptualists - Experimentalists): $-10.7 \pm (2.093)(3.23) = -10.65 \pm 6.75 = \left(-17.40, -3.90\right)$

Factor B (Writer Type):

$$\hat{\mu}_{\bullet 1} = \frac{\sum_{i=1}^{a} \overline{Y}_{i1\bullet}}{a} = \frac{\overline{Y}_{11\bullet} + \overline{Y}_{21\bullet}}{2} = \frac{28.60 + 38.67}{2} = 33.635$$

$$\hat{\mu}_{\bullet 2} = \frac{\sum_{j=1}^{b} \overline{Y}_{i2\bullet}}{b} = \frac{\overline{Y}_{12\bullet} + \overline{Y}_{22\bullet}}{2} = \frac{33.20 + 44.43}{2} = 38.815$$

$$\hat{\mu}_{\bullet 1} - \hat{\mu}_{\bullet 2} = 33.635 - 38.815 = -5.18$$

$$s\left\{\hat{\mu}_{\bullet 1} - \hat{\mu}_{\bullet 2}\right\} = \sqrt{\frac{29.32}{2^2}\left(1^2 + (-1)^2\right)\left(\frac{1}{5} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right)} = \sqrt{10.402} = 3.23$$

$$t\left(0.975, 23 - 4 = 19\right) = 2.093$$

95% CI for $\mu_{\bullet 1} - \mu_{\bullet 2}$ (Poets - Novelists): $-5.18 \pm (2.093)(3.23) = -5.18 \pm 6.75 = \left(-11.93, 1.57\right)$

1-Way Random Effects Model

Treatment Levels in Experiment are Sample from a Population of Levels

Effects are Random Variables (Not Fixed Constants)

$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad i = 1,\ldots,t;\ j = 1,\ldots,n$

$\alpha_i \sim NID\left(0,\sigma_\alpha^2\right) \quad \varepsilon_{ij} \sim NID\left(0,\sigma_\varepsilon^2\right) \quad \left\{\alpha_i\right\},\left\{\varepsilon_{ij}\right\}$ independent

ANOVA obtained as in Fixed Effects Model:

$$SST = n\sum_{i=1}^{t}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \quad df_T = t-1 \quad E\left\{MST\right\} = E\left\{\frac{SST}{t-1}\right\} = \sigma_\varepsilon^2 + n\sigma_\alpha^2$$

$$SSE = \sum_{i=1}^{t}\sum_{j=1}^{n}\left(Y_{ij} - \overline{Y}_{i\bullet}\right)^2 \quad df_E = t\left(n-1\right) \quad E\left\{MSE\right\} = E\left\{\frac{SSE}{t\left(n-1\right)}\right\} = \sigma_\varepsilon^2$$

Estimating Population Mean: $\hat{\mu} = \overline{Y}_{\bullet\bullet} \quad SE\left(\hat{\mu}\right) = \sqrt{\frac{MST}{tn}} \quad (1-\alpha)100\%$ CI: $\hat{\mu} \pm t_{\alpha/2;t-1}\sqrt{\frac{MST}{tn}}$

Point Estimates of Variance Components: $\hat{\sigma}_\varepsilon^2 = MSE \quad \hat{\sigma}_\alpha^2 = \frac{MST - MSE}{n}$

Testing for Trt Effects: $H_0 : \sigma_\alpha^2 = 0 \quad H_A : \sigma_\alpha^2 > 0 \quad TS : F = \frac{MST}{MSE} \quad RR : F \geq F_{\alpha;t-1,t(n-1)}$

Randomized Complete Block Designs – Subjects as Blocks

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} = \mu_i + \beta_j + \varepsilon_{ij} \quad i = 1, \ldots, t;\; j = 1, \ldots, b$$

$$\sum_{i=1}^{t} \alpha_i = 0 \quad E(\varepsilon_{ij}) = 0 \quad V(\varepsilon_{ij}) = \sigma^2 \quad \text{Random Blocks: } \beta_j \sim N\left(0, \sigma_b^2\right)$$

Note:

$$\overline{Y}_{1\bullet} = \frac{1}{b}\left[Y_{11} + \ldots + Y_{1b}\right] = \frac{1}{b}\left[\left(\mu + \alpha_1 + \beta_1 + \varepsilon_{11}\right) + \ldots + \left(\mu + \alpha_1 + \beta_b + \varepsilon_{1b}\right)\right] =$$

$$= \mu + \alpha_1 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{1\bullet}$$

$$\overline{Y}_{2\bullet} = \mu + \alpha_2 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{2\bullet}$$

$$\Rightarrow \quad \overline{Y}_{1\bullet} - \overline{Y}_{2\bullet} = \left(\mu + \alpha_1 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{1\bullet}\right) - \left(\mu + \alpha_2 + \overline{\beta}_{\bullet} + \overline{\varepsilon}_{2\bullet}\right) = \left(\alpha_1 - \alpha_2\right) + \left(\overline{\varepsilon}_{1\bullet} - \overline{\varepsilon}_{2\bullet}\right)$$

- Test for differences among treatment effects:
  - H_0: $\alpha_1 = \ldots = \alpha_t = 0$ $\quad H_A$: Not all $\alpha_i = 0$
  - TS: $F = MST/MSE$ $\quad$ RR: $F \geq F_{\alpha;t-1,(t-1)(b-1)}$

Mixed Effects Models

- Assume:
  - Factor A Fixed (All levels of interest in study)

    $\alpha_1 + \alpha_2 + \ldots + \alpha_a = 0$
  - Factor B Random (Sample of levels used in study)

    $\beta_j \sim N(0,\sigma_b^2)$ (Independent)
  - AB Interaction terms Random

    $(\alpha\beta)_{ij} \sim N(0,\sigma_{ab}^2)$ (Independent)
- Analysis of Variance is computed exactly as in Fixed Effects case (Sums of Squares, df's, MS's)
- Error terms for tests change (See next slide).

Expected Mean Squares for 2-Way ANOVA

Mean Square	df	Fixed Model (A Fixed, B Fixed)	Random Model (A Random, B Random)	Mixed Model (A Fixed, B Random)
MSA	a - 1	$\sigma^2 + \frac{bn\sum_{i=1}^{a}\alpha_i^2}{a-1}$	$\sigma^2 + n\sigma_{\alpha\beta}^2 + bn\sigma_\alpha^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2 + \frac{bn\sum_{i=1}^{a}\alpha_i^2}{a-1}$
MSB	b - 1	$\sigma^2 + \frac{an\sum_{j=1}^{b}\beta_j^2}{b-1}$	$\sigma^2 + n\sigma_{\alpha\beta}^2 + an\sigma_\beta^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2 + an\sigma_\beta^2$
$MSAB$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$	$\sigma^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2$
MSE	$ab(n-1)$	σ^2	σ^2	σ^2

ANOVA Approach – Mixed Effects

Source	df	SS	MS	F
Factor A	a-1	SSA	MSA=SSA/(a-1)	F_A=MSA/MSAB
Factor B	b-1	SSB	MSB=SSB/(b-1)	F_B=MSB/MSAB
Interaction	(a-1)(b-1)	SSAB	MSAB=SSAB/[(a-1)(b-1)]	F_{AB}=MSAB/MSE
Error	ab(n-1)	SSE	MSE=SSE/[ab(n-1)]	
Total	abn-1	TSS		

- Procedure:
  - First test for interaction effects
  - If interaction test not significant, test for Factor A and B effects

Test for Interaction:

$$H_0: \sigma_{ab}^2 = 0$$

$$H_a: \quad \sigma_{ab}^2 > 0$$

$$TS: F_{AB} = \frac{MSAB}{MSE}$$

$$RR: F_{AB} \geq F_{\alpha,(a-1)(b-1),ab(n-1)}$$

Test for Factor A

$$H_0: \alpha_1 = ... = \alpha_a = 0$$

$$H_a: \text{ Not all } \alpha_i = 0$$

$$TS: F_A = \frac{MSA}{MSAB}$$

$$RR: F_A \geq F_{\alpha,(a-1),(a-1)(b-1)}$$

Test for Factor B

$$H_0: \sigma_b^2 = 0$$

$$H_a: \quad \sigma_b^2 > 0$$

$$TS: F_B = \frac{MSB}{MSAB}$$

$$RR: F_B \geq F_{\alpha,(b-1),(a-1)(b-1)}$$

Comparing Main Effects for A (No Interaction)

- Tukey's Method- q in Studentized Range Table with $\nu = (a\text{-}1)(b\text{-}1)$

$$W_{ij}^{A} = q_{\alpha}(a,v)\sqrt{\frac{MSAB}{bn}}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq W_{ij}^{A}$$

$$\text{Tukey's CI: } (\alpha_i - \alpha_j): \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm W_{ij}^{A}$$

- Bonferroni's Method - t-values in Bonferroni table with $\nu = (a\text{-}1)(b\text{-}1)$

$$B_{ij}^{A} = t_{\alpha/2, a(a-1)/2, v}\sqrt{\frac{2MSAB}{bn}}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq B_{ij}^{A}$$

$$\text{Bonferroni's CI:}(\alpha_i - \alpha_j): \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm B_{ij}^{A}$$

Random Effects Models

- Assume:
  - Factor A Random (Sample of levels used in study)

    $\alpha_i \sim N(0,\sigma_a^2)$ (Independent)
  - Factor B Random (Sample of levels used in study)

    $\beta_j \sim N(0,\sigma_b^2)$ (Independent)
  - AB Interaction terms Random

    $(\alpha\beta)_{ij} \sim N(0,\sigma_{ab}^2)$ (Independent)
- Analysis of Variance is computed exactly as in Fixed Effects case (Sums of Squares, df's, MS's)
- Error terms for tests change (See next slide).

ANOVA Approach – Random Effects

Source	df	SS	MS	F
Factor A	a-1	SSA	MSA=SSA/(a-1)	F_A=MSA/MSAB
Factor B	b-1	SSB	MSB=SSB/(b-1)	F_B=MSB/MSAB
Interaction	(a-1)(b-1)	SSAB	MSAB=SSAB/[(a-1)(b-1)]	F_{AB}=MSAB/MSE
Error	ab(n-1)	SSE	MSE=SSE/[ab(n-1)]	
Total	abn-1	TSS		

- Procedure:
  - First test for interaction effects
  - If interaction test not significant, test for Factor A and B effects

Test for Interaction:

$$H_0: \sigma_{ab}^2 = 0$$

$$H_a: \quad \sigma_{ab}^2 > 0$$

$$TS: F_{AB} = \frac{MSAB}{MSE}$$

$$RR: F_{AB} \geq F_{\alpha,(a-1)(b-1),ab(n-1)}$$

Test for Factor A

$$H_0: \sigma_a^2 = 0$$

$$H_a: \quad \sigma_a^2 > 0$$

$$TS: F_A = \frac{MSA}{MSAB}$$

$$RR: F_A \geq F_{\alpha,(a-1),(a-1)(b-1)}$$

Test for Factor B

$$H_0: \sigma_b^2 = 0$$

$$H_a: \quad \sigma_b^2 > 0$$

$$TS: F_B = \frac{MSB}{MSAB}$$

$$RR: F_B \geq F_{\alpha(b-1),(a-1)(b-1)}$$

Nested Designs

- Designs where levels of one factor are nested (as opposed to crossed) wrt other factor
- Examples Include:
  - Classrooms nested within schools
  - Litters nested within Feed Varieties
  - Hair swatches nested within shampoo types
  - Swamps of varying sizes (e.g. large, medium, small)
  - Restaurants nested within national chains

Nested Design - Model

$$Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \varepsilon_{ijk} \quad i = 1,...a \;\; j = 1,...,b_i \;\; k = 1,...,n$$

where:

$Y_{ijk} \equiv$ Response for kth rep of Factor A at ith level, B at jth level within A

$\mu \equiv$ Overall Mean

$\alpha_i \equiv$ Effect of ith level of A (Fixed or Random)

$\beta_{j(i)} \equiv$ Effect of jth level of B within ith level of A (Fixed or Random)

$\varepsilon_{ijk} \equiv$ Random error term for kth rep when A is at i, B is at j(i)

Nested Design - ANOVA

Total Variation:

$$TSS = \sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{...}\right)^2 \qquad df_{Total} = n\sum_{i=1}^{a} b_i - 1$$

Factor A:

$$SSA = n\sum_{i=1}^{a} b_i\left(\overline{Y}_{i..} - \overline{Y}_{...}\right)^2 \qquad df_A = a - 1$$

Factor B Nested Within A

$$SSB(A) = n\sum_{i=1}^{a}\sum_{j=1}^{b_i}\left(\overline{Y}_{ij.} - \overline{Y}_{i..}\right)^2 \qquad df_{B(A)} = \sum_{i=1}^{a} b_i - a$$

Error:

$$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{ij.}\right)^2 \qquad df_E = (n-1)\sum_{i=1}^{a} b_i$$

Estimators, Analysis of Variance, F-tests

$\hat{\mu}_{\bullet\bullet} = \overline{Y}_{\bullet\bullet\bullet} \qquad \hat{\alpha}_i = \overline{Y}_{i\bullet\bullet} - \overline{Y}_{\bullet\bullet\bullet} \qquad \hat{\beta}_{j(i)} = \overline{Y}_{ij\bullet} - \overline{Y}_{i\bullet\bullet}$

Fitted Values: $\hat{Y}_{ijk} = \hat{\mu}_{\bullet\bullet} + \hat{\alpha}_i + \hat{\beta}_{j(i)} = \overline{Y}_{\bullet\bullet\bullet} + \left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{\bullet\bullet\bullet}\right) + \left(\overline{Y}_{ij\bullet} - \overline{Y}_{i\bullet\bullet}\right) = \overline{Y}_{ij\bullet}$

Residuals: $e_{ijk} = Y_{ijk} - \hat{Y}_{ijk} = Y_{ijk} - \overline{Y}_{ij\bullet} \quad \Rightarrow \quad SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(Y_{ijk} - \overline{Y}_{ij\bullet}\right)^2 \qquad df_E = ab(n-1)$

Factor A Sum of Squares: $SSA = bn\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{\bullet\bullet\bullet}\right)^2 \qquad df_A = a-1$

Factor B Within A Sum of Squares: $SSB(A) = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{Y}_{ij\bullet} - \overline{Y}_{i\bullet\bullet}\right)^2 \qquad df_{B(A)} = a(b-1)$

A Fixed, B Fixed

$$E\{MSA\} = \sigma^2 + \frac{bn\sum_{i=1}^{a}\alpha_i^2}{a-1}$$

$$E\{MSB(A)\} = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}\beta_{j(i)}^2}{a(b-1)}$$

$$E\{MSE\} = \sigma^2$$

$H_0: \alpha_1 = \ldots = \alpha_a = 0 \quad TS: F_A = \frac{MSA}{MSE}$

$H_0: \beta_{1(1)} = \ldots = \beta_{b(a)} = 0 \quad TS: F_{B(A)} = \frac{MSB(A)}{MSE}$

A Fixed, B Random

$$E\{MSA\} = \sigma^2 + n\sigma^2_{\beta(\alpha)} + \frac{bn\sum_{i=1}^{a}\alpha_i^2}{a-1}$$

$$E\{MSB(A)\} = \sigma^2 + n\sigma^2_{\beta(\alpha)}$$

$$E\{MSE\} = \sigma^2$$

$H_0: \alpha_1 = \ldots = \alpha_a = 0 \quad TS: F_A = \frac{MSA}{MSB(A)}$

$H_0: \sigma^2_{\beta(\alpha)} = 0 \quad TS: F_{B(A)} = \frac{MSB(A)}{MSE}$

A Random, B Random

$$E\{MSA\} = \sigma^2 + n\sigma^2_{\beta(\alpha)} + bn\sigma^2_{\alpha}$$

$$E\{MSB(A)\} = \sigma^2 + n\sigma^2_{\beta(\alpha)}$$

$$E\{MSE\} = \sigma^2$$

$H_0: \sigma^2_{\alpha} = 0 \quad TS: F_A = \frac{MSA}{MSB(A)}$

$H_0: \sigma^2_{\beta(\alpha)} = 0 \quad TS: F_{B(A)} = \frac{MSB(A)}{MSE}$

Factors A and B Fixed

$$\sum_{i=1}^{a} \alpha_i = 0 \quad \sum_{j=1}^{b_i} \beta_{j(i)} = 0 \; i = 1,\ldots,a \quad \varepsilon_{ijk} \sim N\left(0,\sigma^2\right)$$

Tests for Differences Among Factor A Effects

$H_0 : \alpha_1 = \ldots = \alpha_a = 0 \quad H_A : \text{Not all } \alpha_i = 0$

Test Statistic: $F_A = \dfrac{MSA}{MSE}$ P-value: $P\left(F \geq F_A\right)$

Rejection Region: $F_A \geq F_{\alpha, a-1, (n-1)\sum b_i}$

Tests for Differences Among Factor B Effects

$H_0 : \beta_{j(i)} = 0 \; \forall i, j \quad H_A : \text{Not all } \beta_{j(i)} = 0$

Test Statistic: $F_{B(A)} = \dfrac{MSB(A)}{MSE}$ P-value: $P\left(F \geq F_{B(A)}\right)$

Rejection Region: $F_{B(A)} \geq F_{\alpha, \sum b_i - a, (n-1)\sum b_i}$

Fixed Effects Model (A and B Fixed)

Factor A: $H_0: \alpha_1 = \ldots = \alpha_a = 0 \qquad TS: F_A = \frac{MSA}{MSE} \qquad RR: F_A \geq F\left[1-\alpha; a-1, ab(n-1)\right]$

$$E\left\{\overline{Y}_{i\bullet\bullet}\right\} = \mu_{i\bullet} = \mu_{\bullet\bullet} + \alpha_i \qquad \sigma^2\left\{\overline{Y}_{i\bullet\bullet}\right\} = \frac{\sigma^2}{bn} \qquad s^2\left\{\overline{Y}_{i\bullet\bullet}\right\} = \frac{MSE}{bn}$$

Contrasts among Means of A: $\sum_{i=1}^{a} c_i = 0$

$$L_A = \sum_{i=1}^{a} c_i \mu_{i\bullet} = \sum_{i=1}^{a} c_i \alpha_i \qquad \hat{L}_A = \sum_{i=1}^{a} c_i \overline{Y}_{i\bullet\bullet} \qquad s^2\left\{\hat{L}_A\right\} = \frac{MSE}{bn}\sum_{i=1}^{a} c_i^2$$

$(1-\alpha)100\%$ CI for L_A: $\quad \hat{L}_A \pm t\left(1-(\alpha/2); ab(n-1)\right)\sqrt{\frac{MSE}{bn}\sum_{i=1}^{a} c_i^2}$

All Possible Comparisons among pairs of means of A: $\sum_{i=1}^{a} c_i^2 = 2 \quad C_A = \frac{a(a-1)}{2}$

Tukey: $\left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{i'\bullet\bullet}\right) \pm q\left(1-\alpha; a, ab(n-1)\right)\sqrt{\frac{MSE}{bn}}$ Bonferroni: $\left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{i'\bullet\bullet}\right) \pm t\left(1-\left(\frac{\alpha}{2C_A}\right); ab(n-1)\right)\sqrt{\frac{2MSE}{bn}}$

Factor B(A): $H_0: \beta_{1(1)} = \ldots = \beta_{b(a)} = 0 \qquad TS: F_{B(A)} = \frac{MSB(A)}{MSE} \qquad RR: F_{B(A)} \geq F\left[1-\alpha; a(b-1), ab(n-1)\right]$

$$E\left\{\overline{Y}_{ij\bullet}\right\} = \mu_{ij} = \mu_{\bullet\bullet} + \alpha_i + \beta_{j(i)} \qquad \sigma^2\left\{\overline{Y}_{ij\bullet}\right\} = \frac{\sigma^2}{n} \qquad s^2\left\{\overline{Y}_{ij\bullet}\right\} = \frac{MSE}{n}$$

All Possible Comparisons among pairs of means of B within a given level (i) of A: $C_{B(A)} = \frac{b(b-1)}{2}$

Tukey: $\left(\overline{Y}_{ij\bullet} - \overline{Y}_{ij'\bullet}\right) \pm q\left(1-\alpha; b, ab(n-1)\right)\sqrt{\frac{MSE}{n}}$ Bonferroni: $\left(\overline{Y}_{ij\bullet} - \overline{Y}_{ij'\bullet}\right) \pm t\left(1-\left(\frac{\alpha}{2C_{B(A)}}\right); ab(n-1)\right)\sqrt{\frac{2MSE}{n}}$

Comparing Main Effects for A

- Tukey's Method- q in Studentized Range Table with $\nu = (n\text{-}1)\Sigma b_i$

$$W_{ij}^{A} = q_{\alpha}(a, v)\sqrt{\frac{MSE}{2}\left(\frac{1}{nb_i} + \frac{1}{nb_j}\right)}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq W_{ij}^{A}$$

$$\text{Tukey's CI: } (\alpha_i - \alpha_j) : \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm W_{ij}^{A}$$

- Bonferroni's Method - t-values in Bonferroni table with $\nu = (n\text{-}1)\Sigma b_i$

$$B_{ij}^{A} = t_{\alpha/2, a(a-1)/2, v}\sqrt{MSE\left(\frac{1}{nb_i} + \frac{1}{nb_j}\right)}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq B_{ij}^{A}$$

$$\text{Bonferroni's CI:} (\alpha_i - \alpha_j) \text{: } \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm B_{ij}^{A}$$

Comparing Effects for Factor B Within A

- Tukey's Method- q in Studentized Range Table with $\nu = (n\text{-}1)\Sigma b_i$

$$W^B_{ij(k)} = q_\alpha(b_k, v)\sqrt{\frac{MSE}{n}}$$

$$\text{Conclude: } \beta_{i(k)} \neq \beta_{j(k)} \quad \text{if } \left|\overline{y}_{ki.} - \overline{y}_{kj.}\right| \geq W^B_{ij(k)}$$

$$\text{Tukey's CI: } (\beta_{i(k)} - \beta_{j(k)}) : \left(\overline{y}_{ki.} - \overline{y}_{kj.}\right) \pm W^B_{ij(k)}$$

- Bonferroni's Method - t-values in Bonferroni table with $\nu = (n\text{-}1)\Sigma b_i$

$$B^B_{ij(k)} = t_{\alpha/2, b_k(b_k-1)/2, v}\sqrt{MSE\left(\frac{2}{n}\right)}$$

$$\text{Conclude: } \beta_{i(k)} \neq \beta_{j(k)} \quad \text{if } \left|\overline{y}_{ki.} - \overline{y}_{kj.}\right| \geq B^B_{ij(k)}$$

$$\text{Bonferroni's CI:}(\beta_{i(k)} \text{ - } \beta_{j(k)}): \left(\overline{y}_{ki.} - \overline{y}_{kj.}\right) \pm B^B_{ij(k)}$$

Factor A Fixed and B Random

$$\sum_{i=1}^{a} \alpha_i = 0 \quad \beta_{j(i)} \sim N\left(0, \sigma_b^2\right) \quad \varepsilon_{ijk} \sim N\left(0, \sigma^2\right)$$

Tests for Differences Among Factor A Effects

$$H_0: \alpha_1 = \ldots = \alpha_a = 0 \quad H_A: \text{Not all } \alpha_i = 0$$

$$\text{Test Statistic}: F_A = \frac{MSA}{MSB(A)} \quad \text{P-value}: P\left(F \geq F_A\right)$$

$$\text{Rejection Region}: F_A \geq F_{\alpha, a-1, \sum b_i - a}$$

Tests for Differences Among Factor B Effects

$$H_0: \sigma_b^2 = 0 \ \forall i, j \quad H_A: \sigma_b^2 > 0$$

$$\text{Test Statistic}: F_{B(A)} = \frac{MSB(A)}{MSE} \quad \text{P-value}: P\left(F \geq F_{B(A)}\right)$$

$$\text{Rejection Region}: F_{B(A)} \geq F_{\alpha, \sum b_i - a, (r-1)\sum b_i}$$

Mixed Effects Model (A Fixed and B Random)

Factor A: $H_0: \alpha_1 = \ldots = \alpha_a = 0 \qquad TS: F_A = \frac{MSA}{MSB(A)} \qquad RR: F_A \geq F\left[1-\alpha; a-1, a(b-1)\right]$

$$E\left\{\overline{Y}_{i\bullet\bullet}\right\} = \mu_{i\bullet} = \mu_{\bullet\bullet} + \alpha_i \qquad \sigma^2\left\{\overline{Y}_{i\bullet\bullet}\right\} = \frac{\sigma^2 + n\sigma^2_{\beta(\alpha)}}{bn} \qquad s^2\left\{\overline{Y}_{i\bullet\bullet}\right\} = \frac{MSB(A)}{bn}$$

All Possible Comparisons among pairs of means of A: $\sum_{i=1}^{a} c_i^2 = 2 \quad C_A = \frac{a(a-1)}{2}$

Tukey: $\left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{i'\bullet\bullet}\right) \pm q\left(1-\alpha; a, a(b-1)\right)\sqrt{\frac{MSB(A)}{bn}}$ Bonferroni: $\left(\overline{Y}_{i\bullet\bullet} - \overline{Y}_{i'\bullet\bullet}\right) \pm t\left(1-\left(\frac{\alpha}{2C_A}\right); a(b-1)\right)\sqrt{\frac{2MSB(A)}{bn}}$

Factor B(A): $H_0: \sigma^2_{\beta(\alpha)} = 0 \qquad TS: F_{B(A)} = \frac{MSB(A)}{MSE} \qquad RR: F_{B(A)} \geq F\left[1-\alpha; a(b-1), ab(n-1)\right]$

$$E\left\{MSB(A)\right\} = \sigma^2 + n\sigma^2_{\beta(\alpha)} \qquad E\left\{MSE\right\} = \sigma^2 \qquad \Rightarrow \qquad \left(\frac{1}{n}\right)E\left\{MSB(A)\right\} + \left(-\frac{1}{n}\right)E\left\{MSE\right\} = \sigma^2_{\beta(\alpha)}$$

$\Rightarrow \quad s^2_{\beta(\alpha)} = \frac{1}{n}MSB(A) - \frac{1}{n}MSE$ Approximate df (Satterthwaite): $df_{\beta(\alpha)} = \frac{\left(s^2_{\beta(\alpha)}\right)^2}{\left[\frac{\left(\frac{MSB(A)}{n}\right)^2}{a(b-1)} + \frac{\left(\frac{MSE}{n}\right)^2}{ab(n-1)}\right]}$

Approximate $(1-a)100\%$ CI For $\sigma^2_{\beta(\alpha)}$: $\left(\frac{df_{\beta(\alpha)}s^2_{\beta(\alpha)}}{\chi^2\left(1-\frac{\alpha}{2}; df_{\beta(\alpha)}\right)}, \frac{df_{\beta(\alpha)}s^2_{\beta(\alpha)}}{\chi^2\left(\frac{\alpha}{2}; df_{\beta(\alpha)}\right)}\right)$

Comparing Main Effects for A (B Random)

- Tukey's Method- q in Studentized Range Table with $\nu = \Sigma b_i - a$

$$W_{ij}^{A} = q_{\alpha}(a,v)\sqrt{\frac{MSB(A)}{2}\left(\frac{1}{nb_i}+\frac{1}{nb_j}\right)}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq W_{ij}^{A}$$

$$\text{Tukey's CI: } (\alpha_i - \alpha_j) : \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm W_{ij}^{A}$$

- Bonferroni's Method - t-values in Bonferroni table with $\nu = \Sigma b_i - a$

$$B_{ij}^{A} = t_{\alpha/2, a(a-1)/2, v}\sqrt{MSB(A)\left(\frac{1}{nb_i}+\frac{1}{nb_j}\right)}$$

$$\text{Conclude: } \alpha_i \neq \alpha_j \quad \text{if } \left|\overline{y}_{i..} - \overline{y}_{j..}\right| \geq B_{ij}^{A}$$

$$\text{Bonferroni's CI:}(\alpha_i - \alpha_j): \left(\overline{y}_{i..} - \overline{y}_{j..}\right) \pm B_{ij}^{A}$$

Factors A and B Random

$$\alpha_i \sim N(0, \sigma_a^2) \quad \beta_{j(i)} \sim N(0, \sigma_b^2) \quad \varepsilon_{ijk} \sim N(0, \sigma^2)$$

Tests for Difference s Among Factor A Effects

$$H_0 : \sigma_a^2 = 0 \quad H_A : \sigma_a^2 > 0$$

$$\text{Test Statistic } : F_A = \frac{MSA}{MSB(A)} \quad \text{P - value } : P(F \geq F_A)$$

$$\text{Rejection Region } : F_A \geq F_{\alpha, a-1, \sum b_i - a}$$

Tests for Difference s Among Factor B Effects

$$H_0 : \sigma_b^2 = 0 \ \forall i, j \quad H_A : \sigma_b^2 > 0$$

$$\text{Test Statistic } : F_{B(A)} = \frac{MSB(A)}{MSE} \quad \text{P - value } : P(F \geq F_{B(A)})$$

$$\text{Rejection Region } : F_{B(A)} \geq F_{\alpha, \sum b_i - a, (r-1)\sum b_i}$$

Random Effects Model (A and B Random)

Factor A: $H_0 : \sigma_\alpha^2 = 0 \qquad TS : F_A = \dfrac{MSA}{MSB(A)} \qquad RR : F_A \geq F\left[1-\alpha; a-1, a(b-1)\right]$

$$E\{MSA\} = \sigma^2 + n\sigma_{\beta(\alpha)}^2 + bn\sigma_\alpha^2 \qquad E\{MSB(A)\} = \sigma^2 + n\sigma_{\beta(\alpha)}^2$$

$$\Rightarrow \quad \left(\frac{1}{bn}\right)E\{MSA)\} + \left(-\frac{1}{bn}\right)E\{MSB(A)\} = \sigma_\alpha^2$$

$\Rightarrow \quad s_\alpha^2 = \dfrac{1}{bn}MSA - \dfrac{1}{bn}MSB(A)$ Approximate df (Satterthwaite): $df_A = \dfrac{\left(s_\alpha^2\right)^2}{\left[\dfrac{\left(\dfrac{MSA}{bn}\right)^2}{a-1} + \dfrac{\left(\dfrac{MSB(A)}{bn}\right)^2}{a(b-1)}\right]}$

Approximate $(1-\alpha)100\%$ CI For σ_α^2 : $\left(\dfrac{df_A s_\alpha^2}{\chi^2\left(1-\dfrac{\alpha}{2}; df_A\right)}, \dfrac{df_A s_\alpha^2}{\chi^2\left(\dfrac{\alpha}{2}; df_A\right)}\right)$

Factor B Within A is same as in Mixed Effects Model

Elements of Split-Plot Designs

- Split-Plot Experiment: Factorial design with at least 2 factors, where experimental units wrt factors differ in “size” or “observational points”.
- Whole plot: Largest experimental unit
- Whole Plot Factor: Factor that has levels assigned to whole plots. Can be extended to 2 or more factors
- Subplot: Experimental units that the whole plot is split into (where observations are made)
- Subplot Factor: Factor that has levels assigned to subplots
- Blocks: Aggregates of whole plots that receive all levels of whole plot factor

Split Plot Design

Block 1	Block 2	Block 3	Block 4
A=1, B=1	A=1, B=1	A=1, B=1	A=1, B=1
A=1, B=2	A=1, B=2	A=1, B=2	A=1, B=2
A=1, B=3	A=1, B=3	A=1, B=3	A=1, B=3
A=1, B=4	A=1, B=4	A=1, B=4	A=1, B=4
A=2, B=1	A=2, B=1	A=2, B=1	A=2, B=1
A=2, B=2	A=2, B=2	A=2, B=2	A=2, B=2
A=2, B=3	A=2, B=3	A=2, B=3	A=2, B=3
A=2, B=4	A=2, B=4	A=2, B=4	A=2, B=4
A=3, B=1	A=3, B=1	A=3, B=1	A=3, B=1
A=3, B=2	A=3, B=2	A=3, B=2	A=3, B=2
A=3, B=3	A=3, B=3	A=3, B=3	A=3, B=3
A=3, B=4	A=3, B=4	A=3, B=4	A=3, B=4

Note: Within each block we would assign at random the 3 levels of A to the whole plots and the 4 levels of B to the subplots within whole plots

Examples

- Agriculture: Varieties of a crop or gas may need to be grown in large areas, while varieties of fertilizer or varying growth periods may be observed in subsets of the area.
- Engineering: May need long heating periods for a process and may be able to compare several formulations of a by-product within each level of the heating factor.
- Behavioral Sciences: Many studies involve repeated measurements on the same subjects and are analyzed as a split-plot (See Repeated Measures lecture)

Design Structure

- Blocks: *b* groups of experimental units to be exposed to all combinations of whole plot and subplot factors
- Whole plots: *a* experimental units to which the whole plot factor levels will be assigned to at random within blocks
- Subplots: *c* subunits within whole plots to which the subplot factor levels will be assigned to at random.
- Fully balanced experiment will have $n=abc$ observations

Data Elements (Fixed Factors, Random Blocks)

- Y_{ijk}: Observation from wpt i, block j, and spt k
- μ : Overall mean level
- α_i : Effect of i^{th} level of whole plot factor (Fixed)
- b_j: Effect of j^{th} block (Random)
- $(ab)_{ij}$: Random error corresponding to whole plot elements in block j where wpt i is applied
- γ_k: Effect of k^{th} level of subplot factor (Fixed)
- $(\alpha\gamma)_{ik}$: Interaction btwn wpt i and spt k
- $(bc)_{jk}$: Interaction btwn block j and spt k (often set to 0)
- ε_{ijk}: Random Error= $(bc)_{jk}$+ $(abc)_{ijk}$
- Note that if block/spt interaction is assumed to be 0, ε represents the block/spt within wpt interaction

Model and Common Assumptions

- $Y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + \gamma_k + (\alpha\gamma)_{ik} + \varepsilon_{ijk}$

$$\sum_{i=1}^{a} \alpha_i = 0$$

$$b_j \sim NID(0, \sigma_b^2)$$

$$(ab)_{ij} \sim NID(0, \sigma_{ab}^2)$$

$$\sum_{k=1}^{c} \gamma_k = 0$$

$$\sum_{i=1}^{a} (\alpha\gamma)_{ik} = \sum_{k=1}^{c} (\alpha\gamma)_{ik} = 0$$

$$\varepsilon_{ijk} \sim NID(0.\sigma_\varepsilon^2)$$

$$COV(b_j, (ab)_{ij}) = COV(b_j, \varepsilon_{ijk}) = COV((ab)_{ij}, \varepsilon_{ijk}) = 0$$

Tests for Fixed Effects

$$\text{Whole Plot Trt Effects: } H_0 : \alpha_1 = \cdots = \alpha_a = 0$$

$$\text{Test Statistic: } F_{WP} = \frac{MS_{WP}}{MS_{BLOCK*WP}}$$

$$P_{WP} = P(F \geq F_{WP} \mid F \sim F_{a-1.(a-1)(b-1)})$$

$$\text{Subplot Trt Effects: } H_0 : \gamma_1 = \cdots = \gamma_c = 0$$

$$\text{Test Statistic: } F_{SP} = \frac{MS_{SP}}{MS_{ERROR}}$$

$$P_{SP} = P(F \geq F_{SP} \mid F \sim F_{c-1.a(b-1)(c-1)})$$

$$\text{WP} \times \text{SP Interaction: } H_0 : (\alpha\gamma)_{ik} = 0 \ \forall i, k$$

$$\text{Test Statistic: } F_{WP \times SP} = \frac{MS_{WP \times SP}}{MS_{ERROR}}$$

$$P_{WP \times SP} = P(F \geq F_{WP \times SP} \mid F \sim F_{(a-1)(c-1).a(b-1)(c-1)})$$

Comparing Factor Levels

Whole Plot Factor Levels :

$$95\%\ \text{CI for } (\alpha_i - \alpha_{i'}): \quad \left(\overline{Y}_{i..} - \overline{Y}_{i'..}\right) \pm t \sqrt{\frac{2MS_{BLOCK \times WP}}{bc}}$$

Sub Plot Factor Levels :

$$95\%\ \text{CI for } (\gamma_k - \gamma_{k'}): \quad \left(\overline{Y}_{..k} - \overline{Y}_{..k'}\right) \pm t \sqrt{\frac{2MS_{ERROR}}{ab}}$$

Sub Plot Effects Within same whole plot (Interaction) :

$$95\%\ \text{CI for } (\gamma_k - \gamma_{k'}) + \left((\alpha\gamma)_{ik} - (\alpha\gamma)_{ik'}\right): \quad \left(\overline{Y}_{i.k} - \overline{Y}_{i.k'}\right) \pm t \sqrt{\frac{2MS_{ERROR}}{b}}$$

Whole Plot Effects within same sub plot (Interaction) :

$$\left(\overline{Y}_{i.k} - \overline{Y}_{i'.k}\right) \pm t \sqrt{\frac{2\left[MS_{BLOCK \times WP} + (c-1)MS_{ERROR}\right]}{bc}} \qquad \text{(df given below)}$$

$$\hat{\nu} = \frac{\left[(c-1)MS_{ERROR} + MS_{BLOCK \times WP}\right]^2}{\left[\frac{\left[(c-1)MS_{ERROR}\right]^2}{a(b-1)(c-1)} + \frac{\left[MS_{BLOCK \times WP}\right]^2}{(a-1)(b-1)}\right]}$$

Repeated Measures Designs

- *a* Treatments/Conditions to compare
- *N* subjects to be included in study (each subject will receive only one treatment)
  - *n* subjects receive trt *i*: $an = N$
- *t* time periods of data will be obtained
- Effects of trt, time and trtxtime interaction of primary interest.
  - Between Subject Factor: Treatment
  - Within Subject Factors: Time, TrtxTime

Model

$$Y_{ijk} = \mu + \alpha_i + b_{j(i)} + \tau_k + (\alpha\tau)_{ik} + \varepsilon_{ijk}$$

$\mu \equiv$ overall mean

$\alpha_i \equiv$ effect of trt $i \quad \sum_{i=1}^{a} \alpha_i = 0$

$b_{j(i)} \equiv$ effect of j^{th} subject in trt $i \quad b_{j(i)} \sim NID(0, \sigma_b^2)$

$\tau_k \equiv$ effect of k^{th} time period $\quad \sum_{k=1}^{t} \tau_k = 0$

$(\alpha\tau)_{ik} \equiv$ interaction between trt i and time $k \quad \sum_{i=1}^{a} (\alpha\tau)_{ik} = \sum_{k=1}^{t} (\alpha\tau)_{ik} = 0$

$\varepsilon_{ijk} \equiv$ random error term $\quad \varepsilon_{ijk} \sim NID(0, \sigma_\varepsilon^2)$

Note the random error term is actually the interaction between subjects (within treatments) and time

Tests for Fixed Effects

$$\text{Treatment Effects}: \; H_0: \alpha_1 = \cdots = \alpha_a = 0$$

$$\text{Test Statistic}: \; F_{TRTS} = \frac{MS_{TRTS}}{MS_{SUBJECTS(TRTS)}}$$

$$P_{TRTS} = P(F \geq F_{TRTS} \mid F \sim F_{a-1,a(n-1)})$$

$$\text{Time Effects}: \; H_0: \tau_1 = \cdots = \tau_t = 0$$

$$\text{Test Statistic}: \; F_{TIME} = \frac{MS_{TIME}}{MS_{ERROR}}$$

$$P_{TIME} = P(F \geq F_{TIME} \mid F \sim F_{t-1,a(n-1)(t-1)})$$

$$\text{Treatment/Time Interaction}: \; H_0: (\alpha\tau)_{ik} = 0 \; \forall i,k$$

$$\text{Test Statistic}: \; F_{TRT\times TIME} = \frac{MS_{TRT\times TIME}}{MS_{ERROR}}$$

$$P_{TRT\times TIME} = P(F \geq F_{TRT\times TIME} \mid F \sim F_{(a-1)(t-1),a(n-1)(t-1)})$$

Comparing Factor Levels

Comparing Treatment Levels :

95% CI for $\alpha_i - \alpha_{i'}$: $\left(\overline{Y}_{i..} - \overline{Y}_{i'..}\right) \pm t\sqrt{\dfrac{2MS_{SUBJECTS(TRTS)}}{nt}}$

Comparing Time Levels :

95% CI for $\tau_{\mathrm{k}} - \tau_{k'}$: $\left(\overline{Y}_{..k} - \overline{Y}_{..k'}\right) \pm t\sqrt{\dfrac{2MS_{ERROR}}{an}}$

Comparing Treatment Levels Within Time Levels :

$$\left(\overline{Y}_{i.k} - \overline{Y}_{i'.k}\right) \pm t\sqrt{\frac{2\left(MS_{SUBJECTS(TRTS)} + (t-1)MS_{ERROR}\right)}{nt}}$$

with approximate df :

$$\hat{\nu} = \frac{\left[(t-1)MS_{ERROR} + MS_{SUBJECT(TRT)}\right]^2}{\left[\dfrac{\left[(t-1)MS_{ERROR}\right]^2}{a(n-1)(t-1)} + \dfrac{\left[MS_{SUBJECT(TRT)}\right]^2}{a(n-1)}\right]}$$