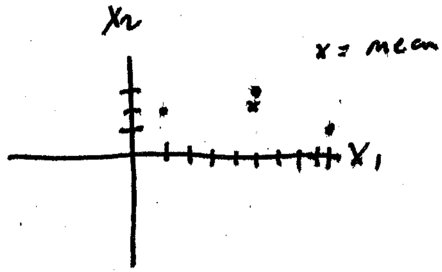
Chapter 3 Problems

3.1 $X = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad \bar{X} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

a)



b) deviation vectors: $\underset{\sim}{d_1} = \begin{bmatrix} 9 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$ $\quad \underset{\sim}{d_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

c) $L_{d_1} = \sqrt{4^2 + 0^2 + (-4)^2} = \sqrt{32} = 5.6569$

$L_{d_2} = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2} = 1.4142$

$$\cos\theta = \frac{\underset{\sim}{d_1}'\underset{\sim}{d_2}}{L_{d_1} L_{d_2}} = \frac{4(-1) + 0(1) + (-4)(0)}{\sqrt{32}\,\sqrt{2}} = \frac{-4}{8} = -\frac{1}{2}$$

$\Rightarrow \theta = \cos^{-1}(-\tfrac{1}{2}) = 120°$

3.3 $n=3$ $\quad X = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 1 & 6 \\ 4 & 0 & 4 \end{bmatrix}$ $\quad \bar{X}_1 = \frac{1}{3}\begin{bmatrix} 1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$

$\underset{\sim}{y_1} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$ $\quad \bar{X}_1 \underset{\sim}{1} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ $\quad \underset{\sim}{y_1} - \bar{X}_1 \underset{\sim}{1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

3.4 $\underset{\sim}{y}_1 = \begin{bmatrix} 3.5 \\ 2.5 \\ 1.8 \\ 1.7 \\ 1.6 \\ 1.5 \end{bmatrix}$ $\bar{X}_1 = 2.1$

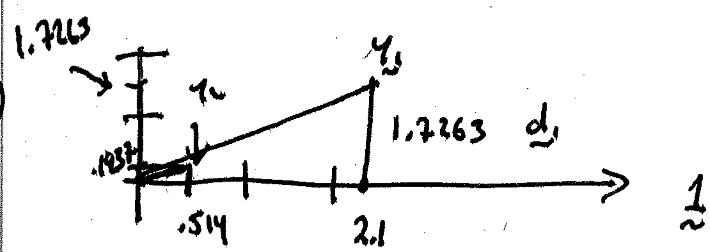
a) Projection $= \dfrac{y_1' \underset{\sim}{1}}{\underset{\sim}{1}'\underset{\sim}{1}} \underset{\sim}{1} = 2.1 \underset{\sim}{1} = \bar{x}_1 \underset{\sim}{1}$

b) $\underset{\sim}{d}_1 = \underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1} = \begin{bmatrix} 1.4 \\ 0.4 \\ -0.3 \\ -0.4 \\ -0.5 \\ -0.6 \end{bmatrix}$

$L_{\underset{\sim}{d}_1} = \sqrt{\underset{\sim}{d}_1' \underset{\sim}{d}_1} = \sqrt{1.4^2 + 0.4^2 + (-0.3)^2 + (-0.4)^2 + (-0.5)^2 + (-0.6)^2}$

$= \sqrt{1.96 + .16 + .09 + .16 + .25 + .36} = \sqrt{2.98} = 1.7263$

$S = \sqrt{\dfrac{\sum (x_{i1} - \bar{x})^2}{n-1}} \Rightarrow \text{Length} = \sqrt{n-1}\, S$

c) 

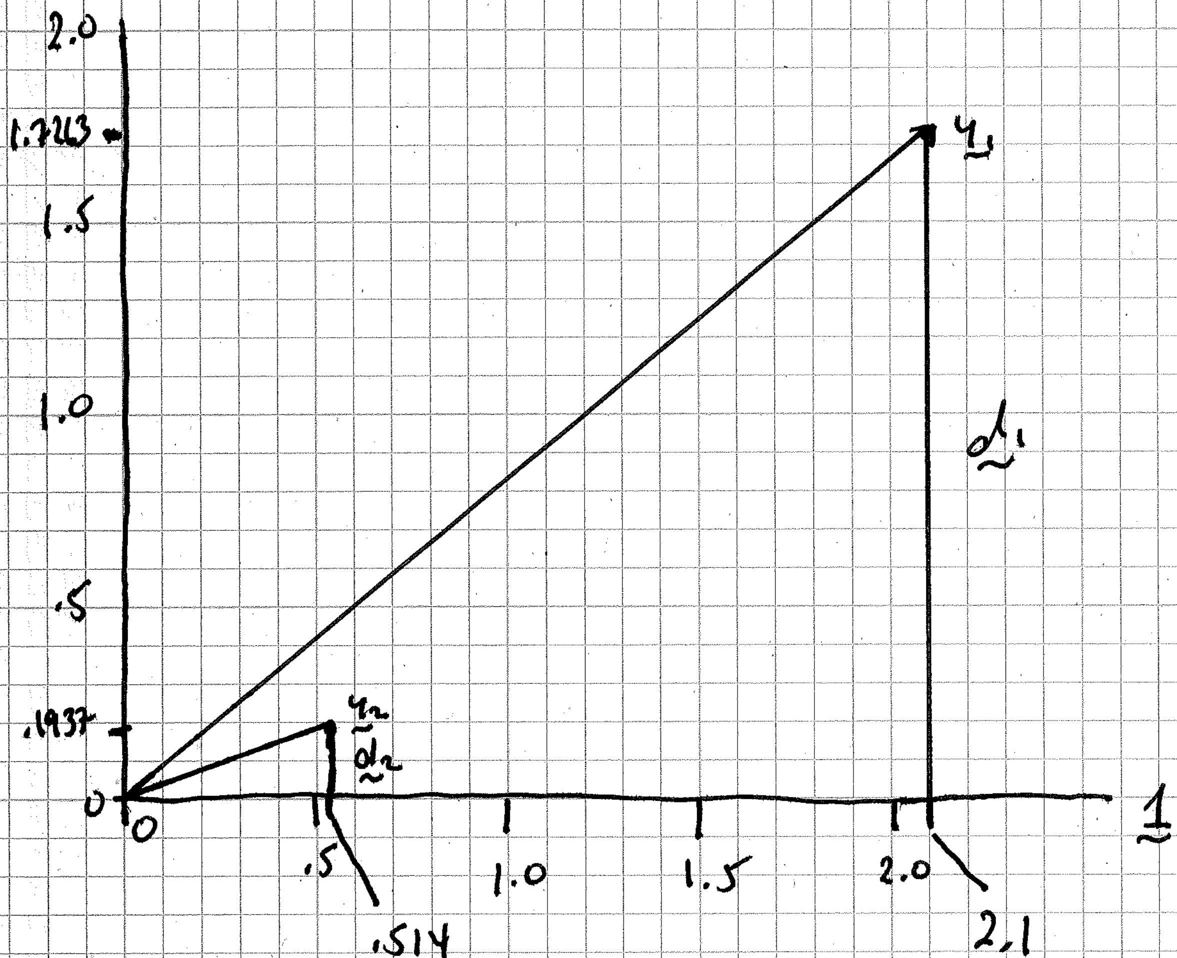
d) $\underset{\sim}{y}_2 = \begin{bmatrix} .623 \\ .593 \\ .512 \\ .500 \\ .463 \\ .395 \end{bmatrix}$ $\bar{X}_2 = .514$ $\underset{\sim}{d}_2 = \begin{bmatrix} .109 \\ .079 \\ -.002 \\ -.014 \\ -.071 \\ -.119 \end{bmatrix}$

$L_{\underset{\sim}{d}_2} = \sqrt{\underset{\sim}{d}_2' \underset{\sim}{d}_2} = \sqrt{.0375} = .1937$

3.3

3.5(e) $\quad L_{\underset{\sim}{d}_1} = 1.7263 \quad L_{\underset{\sim}{d}_2} = .1937$

$\bar{x}_1 = 2.1 \quad \bar{x}_2 = .514$



$\underset{\sim}{d}_1' \underset{\sim}{d}_2 = L_{d_1} L_{d_2} \cos\theta_{12}$

$d_1' d_2 = 1.4(.109) + 0.4(.079) + (-.3)(-.002) + (-0.4)(-.014) + (-0.5)(-.071) + (-0.6)(-.119)$

$= .2973 \Rightarrow \cos\theta_{12} = \frac{.2973}{1.7263(.1937)} = .8891$

$\Rightarrow \theta_{12} = \cos^{-1}(.8891) = 27.24$

3.6 $X = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ $\qquad \underset{\sim}{\bar{X}} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

a)

$$\underset{\sim}{1}\underset{\sim}{\bar{X}}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [2 \quad 3 \quad 1] = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$X - \underset{\sim}{1}\underset{\sim}{\bar{X}}' = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

row3 + row2 + row1 = 0 => Not full rank

col 1 + col 2 − col 3 = 0 => " " "

b)

$$(X - \underset{\sim}{1}\underset{\sim}{\bar{X}}')'(X - \underset{\sim}{1}\underset{\sim}{\bar{X}}') = \begin{bmatrix} 18 & -3 & 15 \\ -3 & 2 & -1 \\ 15 & -1 & 14 \end{bmatrix}$$

$$S = \frac{1}{n-1}(X - \underset{\sim}{1}\underset{\sim}{\bar{X}}')'(X - \underset{\sim}{1}\underset{\sim}{\bar{X}}') = \begin{bmatrix} 9 & -1.5 & 7.5 \\ -1.5 & 1 & -0.5 \\ 7.5 & -0.5 & 7 \end{bmatrix}$$

$$|S| = [9(1)(7) + (-1.5)(-0.5)(7.5) + 7.5(-1.5)(-0.5)]$$
$$- [7.5(1)(7.5) + (-1.5)(-1.5)(7) + 9(-.5)(.5)]$$
$$= (63 + 5.625 + 5.625) - (56.25 + 15.75 + 2.25)$$
$$= 74.25 - 74.25 = 0$$ => Linear dependent deviations

3.5

3.6 c) Total Sample Variance $= 9+1+7 = 17$

3.8 $S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $S_2 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

a) Total Variance $= 1+1+1 = 3$ for S_1 S_2

b) $|S_1| = 1$

$|S_2| = \left(1 - \frac{1}{8} - \frac{1}{8}\right) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) = \frac{3}{4} - \frac{3}{4} = 0$

Linear dependencies among deviation vectors for S_2.

3.11 $S = \begin{bmatrix} 252.04 & -68.43 \\ -68.43 & 123.67 \end{bmatrix}$ $D^{1/2} = \begin{bmatrix} 15.8758 & 0 \\ 0 & 11.1207 \end{bmatrix}$

$D^{-1/2} = \begin{bmatrix} .0630 & 0 \\ 0 & .0899 \end{bmatrix}$

$D^{-1/2} S D^{-1/2} = \begin{bmatrix} 15.8758 & -4.3111 \\ -6.1519 & 11.1179 \end{bmatrix} \begin{bmatrix} .0630 & 0 \\ 0 & .0899 \end{bmatrix}$

$= \begin{bmatrix} 1.0002 & -.3876 \\ -.3876 & 0.9995 \end{bmatrix} \approx \begin{bmatrix} 1 & -.3876 \\ -.3876 & 1 \end{bmatrix} = R$

3.11 Continued

$$D^{1/2} R D^{1/2} = \begin{bmatrix} 15.8758 & -6.1535 \\ -4.3104 & 11.1207 \end{bmatrix} \begin{bmatrix} 15.8758 & 0 \\ 0 & 11.1207 \end{bmatrix}$$

$$= \begin{bmatrix} 252.04 & -68.43 \\ -68.43 & 123.67 \end{bmatrix} = S$$

3.13 Given X and sample correlation matrix R

$$z_{jk} = \frac{x_{jk} - \bar{x}_k}{\sqrt{s_{kk}}}$$

$$\sum_{j=1}^{n} z_{jk}^2 = \sum_{j=1}^{n} \frac{(x_{jk} - \bar{x}_k)^2}{s_{kk}} = \frac{1}{s_{kk}} (n-1) s_{kk} = n-1$$

$$k \neq i \quad \sum_{j=1}^{n} z_{jk} z_{ji} = \sum_{j=1}^{n} \frac{(x_{jk} - \bar{x}_k)(x_{ji} - \bar{x}_i)}{\sqrt{s_{kk}}\sqrt{s_{ii}}} = \frac{1}{\sqrt{s_{kk}}\sqrt{s_{ii}}} (n-1) s_{ik}$$

$$= (n-1) r_{ik}$$

$$\Rightarrow S_z = \frac{1}{n-1} \begin{bmatrix} n-1 & (n-1) r_{12} & \cdots & (n-1) r_{1p} \\ \vdots & & & \\ (n-1) r_{1p} & (n-1) r_{2p} & \cdots & n-1 \end{bmatrix} = R$$

3.14

$$X = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} \end{array} \qquad c'X = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$b'X = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

a)

$\underset{\sim}{c}'\underset{\sim}{x}_1 = -9 + 2 = -7 \quad \underset{\sim}{c}'\underset{\sim}{x}_2 = -5 + 6 = 1 \quad \underset{\sim}{c}'\underset{\sim}{x}_3 = -1 + 4 = 3$

$\underset{\sim}{b}'\underset{\sim}{x}_1 = 18 + 3 = 21 \quad \underset{\sim}{b}'\underset{\sim}{x}_2 = 10 + 9 = 19 \quad \underset{\sim}{b}'\underset{\sim}{x}_3 = 2 + 6 = 8$

Mean of $\underset{\sim}{c}'\underset{\sim}{X}$: $\frac{1}{3}(-7 + 1 + 3) = -1$

Mean of $\underset{\sim}{b}'\underset{\sim}{X} = \frac{1}{3}(21 + 19 + 8) = 16$

Variance of $\underset{\sim}{c}'\underset{\sim}{X}$: $\frac{1}{3-1}\left[(-7-(-1))^2 + (1-(-1))^2 + (3-(-1))^2\right]$

$= \frac{1}{2}\left[36 + 4 + 16\right] = \frac{1}{2}(56) = 28$

Variance of $\underset{\sim}{b}'\underset{\sim}{X}$: $\frac{1}{3-1}\left[(21-16)^2 + (19-16)^2 + (8-16)^2\right]$

$= \frac{1}{2}\left[25 + 9 + 64\right] = \frac{98}{2} = 49$

Covariance of $c'X, b'X$:

$\frac{1}{3-1}\left[(-7-(-1))(21-16) + (1-(-1))(19-16) + (3-(-1))(8-16)\right]$

$= \frac{1}{2}\left[-30 + 6 - 32\right] = \frac{-56}{2} = -28$

b) $\underset{\sim}{\bar{X}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \underset{\sim}{d}_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \quad \underset{\sim}{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$S = \frac{1}{3-1} \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$$

3.14 b) Continued

Mean of $\underset{\sim}{c}'X$: $\underset{\sim}{c}'\bar{X} = \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = -5+4 = 1$

Mean of $\underset{\sim}{b}'X$: $\underset{\sim}{b}'\bar{X} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2(5)+3(2) = 16$

Variance of $\underset{\sim}{c}'X$: $\underset{\sim}{c}'S\underset{\sim}{c} = \begin{bmatrix} -20 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 20+8 = 28$

Variance of $\underset{\sim}{b}'X$: $\underset{\sim}{b}'S\underset{\sim}{b} = \begin{bmatrix} 26 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 52-3 = 49$

Cov of $\underset{\sim}{c}'X, \underset{\sim}{b}'X$: $\underset{\sim}{c}'S\underset{\sim}{b} = \begin{bmatrix} -20 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -40+12 = -28$

3.16 $V \equiv$ random vector w/ $E\{V\} = \mu_v$

$E\{(V-\mu_v)(V-\mu_v)'\} = \Sigma_v$ obtain $E\{VV'\}$

$$\Sigma_v = E\{VV' - V\mu_v' - \mu_v V' + \mu_v \mu_v'\}$$

$$= E\{VV'\} - \mu_v\mu_v' - \mu_v\mu_v' + \mu_v\mu_v' = E\{VV'\} - \mu_v\mu_v'$$

$$\Rightarrow E\{VV'\} = \Sigma_v + \mu_v\mu_v'$$

3.18 X_1 = petroleum X_2 = nat gas X_3 = hydroelec X_4 = nuclear

$$\bar{X} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix} \quad S = \begin{bmatrix} 0.856 & .635 & .173 & .096 \\ .635 & .568 & .128 & .067 \\ .173 & .128 & .171 & .039 \\ .096 & .067 & .039 & .043 \end{bmatrix}$$

a) $c' = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

$\underset{\sim}{c}'\underset{\sim}{\bar{X}} = .766 + .508 + .438 + .161 = 1.873$

$\underset{\sim}{c}'S\underset{\sim}{c} = \sum_{i=1}^{4}\sum_{j=1}^{4} s_{ij} = .856 + 2(.635) + 2(.173) + 2(.096) + .568$
$+ 2(.128) + 2(.067) + .171 + 2(.039) + .043$

$= 3.914$

b) $\underset{\sim}{b}' = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$

$\underset{\sim}{b}'\underset{\sim}{\bar{X}} = .766 - .508 = .158$

$\underset{\sim}{b}'\underset{\sim}{S}\underset{\sim}{c}$ $\begin{bmatrix}(.856 - .635) = .221 & .067 & .045 & .029\end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$= .221 - .067 = .154$

$\underset{\sim}{c}'\underset{\sim}{S}\underset{\sim}{b} = (.856 + .635 + .173 + .096) - (.635 + .568 + .128 + .067)$
$= 1.760 - 1.398 = 0.362$

Problem 3.20

R Program

```
prob3.20 <- read.table("http://www.stat.ufl.edu/~winner/sta4702/data/wichern/T3-2.DAT",
    header=F, col.names=c("x1","x2"))
attach(prob3.20)

X <- cbind(x1,x2)
n <- nrow(X)
I_n <- diag(n)
n_1 <- rep(1,n)
J_n <- matrix(rep(1,n^2),ncol=n)

(xbar <- (1/n) * t(X) %*% n_1)
(S <- (1/(n-1)) * (t(X) %*% (I_n - (1/n)*J_n) %*% X))

c <- matrix(c(-1,1),ncol=1)

(cxbar <- t(c) %*% xbar)
(cSc <- t(c) %*% S %*% c)

dx1x2 <- x2-x1; mean(dx1x2); var(dx1x2)
```

R Output

```
> (xbar <- (1/n) * t(X) %*% n_1)
     [,1]
x1  9.420
x2 19.272
> (S <- (1/(n-1)) * (t(X) %*% (I_n - (1/n)*J_n) %*% X))
         x1       x2
x1 14.13917 13.47267
x2 13.47267 62.23877
>
> c <- matrix(c(-1,1),ncol=1)
>
> (cxbar <- t(c) %*% xbar)
      [,1]
[1,] 9.852
> (cSc <- t(c) %*% S %*% c)
        [,1]
[1,] 49.4326
>
> dx1x2 <- x2-x1; mean(dx1x2); var(dx1x2)
[1] 9.852
[1] 49.4326
```