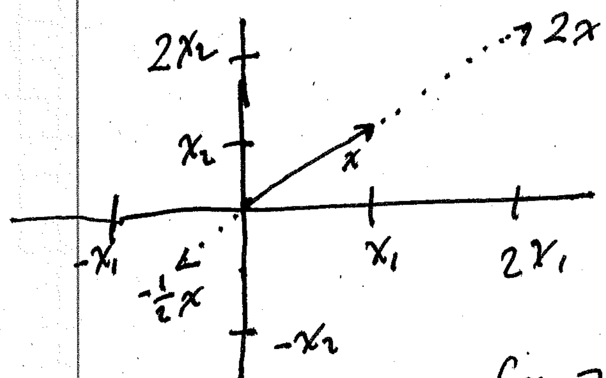
Chapter 2

2.2. Basics of Matrix and Vector Algebra

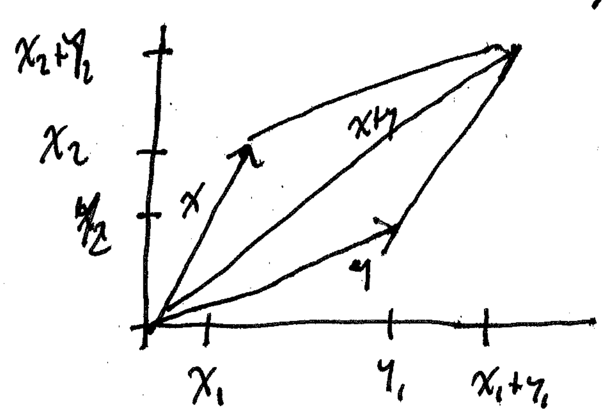
$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \underset{\sim}{x}' = [x_1 \ x_2 \ \dots \ x_n]$$

Transpose

Expanded / Contracted
$$c\underset{\sim}{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$



Addition
$$\underset{\sim}{x} + \underset{\sim}{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$



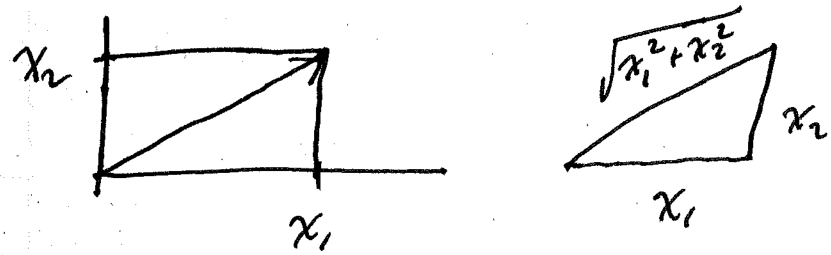
Length: $L_x = \sqrt{x_1^2 + x_2^2}$ $(n=2)$

Generally: $L_x = \sqrt{x_1^2 + \dots + x_n^2}$

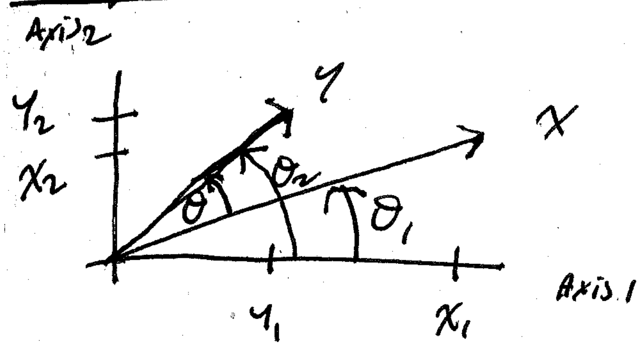
$$L_{cx} = \sqrt{c^2x_1^2 + \dots + c^2x_n^2} = |c|\sqrt{x_1^2+\dots+x_n^2} = |c|\,L_x$$

$|c| > 1 \Rightarrow$ Expanded $\quad 0 < |c| < 1 \Rightarrow$ Contracted

$c = L_x^{-1} \Rightarrow cx = L_x^{-1} \underset{\sim}{x}$ has unit length in direction of x



Angle



$\theta = \theta_2 - \theta_1$

$$\cos(\theta_1) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x_1}{L_x} \qquad \cos\theta_2 = \frac{y_1}{L_y}$$

$$\sin\theta_1 = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x_2}{L_x} \qquad \sin\theta_2 = \frac{y_2}{L_y}$$

$$\cos\theta = \cos(\theta_2 - \theta_1) = \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1$$

$$= \left(\frac{y_1}{L_y}\right)\left(\frac{x_1}{L_x}\right) + \left(\frac{y_2}{L_y}\right)\left(\frac{x_2}{L_x}\right) = \frac{x_1 y_1 + x_2 y_2}{L_x L_y}$$

<u>Inner Product</u> $\quad \underset{\sim}{x}'\underset{\sim}{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2$

$$L_x = \sqrt{\underset{\sim}{x}'\underset{\sim}{x}} \Rightarrow \cos\theta = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{\sqrt{\underset{\sim}{x}'\underset{\sim}{x}}\sqrt{\underset{\sim}{y}'\underset{\sim}{y}}} = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{L_x L_y}$$

$\cos(90°) = \cos(270°) = 0 \qquad \cos\theta = 0 \iff \underset{\sim}{x}'\underset{\sim}{y} = 0$

$\Rightarrow \underset{\sim}{x}, \underset{\sim}{y}$ perpendicular when $\underset{\sim}{x}'\underset{\sim}{y} = 0$

General # of dimensions n: $\underset{\sim}{x}'\underset{\sim}{y} = x_1 y_1 + \cdots + x_n y_n = \underset{\sim}{y}'\underset{\sim}{x}$

$$\cos\theta = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{L_x L_y} = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{\sqrt{\underset{\sim}{x}'\underset{\sim}{x}}\sqrt{\underset{\sim}{y}'\underset{\sim}{y}}}$$

EXAMPLE 2.1

$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad \underset{\sim}{y} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \qquad 3\underset{\sim}{x} = ?$

$\underset{\sim}{x} + \underset{\sim}{y} = ?$

$3\underset{\sim}{x} = 3\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix} \qquad \underset{\sim}{x} + \underset{\sim}{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$\underset{\sim}{x}'\underset{\sim}{x} = 1^2 + 3^2 + 2^2 = 14 \qquad \underset{\sim}{y}'\underset{\sim}{y} = (-2)^2 + 1^2 + (-1)^2 = 6$

$\underset{\sim}{x}'\underset{\sim}{y} = 1(-2) + 3(1) + 2(-1) = -1$

$\Rightarrow L_x = \sqrt{\underset{\sim}{x}'\underset{\sim}{x}} = \sqrt{14} = 3.742 \qquad L_y = \sqrt{\underset{\sim}{y}'\underset{\sim}{y}} = \sqrt{6} = 2.449$

$$\cos\theta = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{L_x L_y} = \frac{-1}{3.742(2.449)} = -.1091$$

$\Rightarrow \theta = \cos^{-1}(-.1091) = 96.264°$

Ex. 2.1 continued

$$L_{3x} = \sqrt{3^2+9^2+6^2} = \sqrt{126} = \sqrt{14(9)} = 3\sqrt{14} = 3L_x$$

Linearly Dependent $\underset{\sim}{x}, \underset{\sim}{y}$

$c_1 \underset{\sim}{x} + c_2 \underset{\sim}{y} = \underset{\sim}{0}$ for some c_1, c_2 (not both 0)

Linearly dependent $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k$

$\exists\ c_1, \ldots, c_k$ (not all 0) $\ni c_1 \underset{\sim}{x}_1 + \cdots + c_k \underset{\sim}{x}_k = \underset{\sim}{0}$

Example 2.2 - Indentifying Linearly Indep. Vectors

$$\underset{\sim}{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \underset{\sim}{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + c_3 \underset{\sim}{x}_3 = \underset{\sim}{0} \Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 \qquad - 2c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0 \end{aligned}$$

Only solution is $c_1 = c_2 = c_3 = 0$

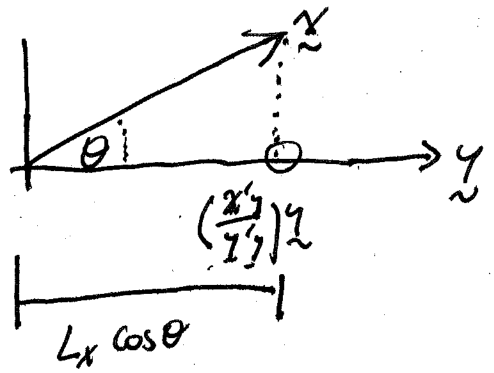
would need
$c_1 = c_3 \neq 0$
$2c_1 + c_2 = 2c_1 - c_2$
$\Rightarrow c_2 = -c_2 \neq 0$

Projection of $\underset{\sim}{x}$ on $\underset{\sim}{y}$

$$= \frac{\underset{\sim}{x}'\underset{\sim}{y}}{\underset{\sim}{y}'\underset{\sim}{y}} \underset{\sim}{y} = \frac{\underset{\sim}{x}'\underset{\sim}{y}}{L_y} \left(\frac{1}{L_y}\right) \underset{\sim}{y} \qquad \left(\frac{1}{L_y} \underset{\sim}{y} \text{ has unit length}\right)$$

Length of Projection $= \dfrac{|\underset{\sim}{x}'\underset{\sim}{y}|}{L_y} = L_x \left| \dfrac{\underset{\sim}{x}'\underset{\sim}{y}}{L_x L_y} \right| = L_x |\cos\theta|$

Projection of $\underset{\sim}{x}$ on $\underset{\sim}{y}$



Matrices

$$\underset{(n\times p)}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix}$$

Transpose $\underset{(p\times n)}{A'}$

Scalar multiple cA

Sums and Scalar multiples

Multiplication: $\underset{(n\times k)}{A}\ \underset{(k\times p)}{B} \equiv$ $n\times p$ w/ elements being inner products of rows of A w/ columns of B

Square matrices

Symmetric $\Rightarrow$ $A = A'$ or $a_{ij} = a_{ji}$ $\forall\ i,j$

A, B of same dimension $\Rightarrow$ AB, BA exist, not necessarily that $AB = BA$

$AA^{-1} = I = A^{-1}A$

- Identity matrix
- Inverse matrix when columns (rows) of A are lin. indep.

Square matrices continued

Orthogonal Matrices

$$QQ' = Q'Q = I \implies Q' = Q^{-1}$$

i^{th} row of $Q \equiv \underset{\sim}{q}_i'$ $\quad QQ' = I$

$\implies q_i'q_i = 1, \quad q_i'q_j = 0 \quad \forall\ i \neq j$

Rows have unit length $(q_i'q_i = 1)$ and are mutually perpendicular (orthogonal) $(\underset{\sim}{q}_i'\underset{\sim}{q}_j = 0)$.

Eigenvectors and eigenvalues

A has eigenvalue λ, w/ corresponding eigenvector $\underset{\sim}{x} \neq \underset{\sim}{0}$

if: $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ $\quad$ $\underset{\sim}{x}$ normalized s.t. $\underset{\sim}{x}'\underset{\sim}{x} = 1$

Notation: Normalized eigenvectors $\equiv \underset{\sim}{e}$

$A \equiv$ Square, symmetric $k \times k$ matrix, A has k pairs of eigenvalues & eigenvectors:

$$\lambda_1, \underset{\sim}{e}_1 \quad \lambda_2, \underset{\sim}{e}_2 \quad \ldots \quad \lambda_k, \underset{\sim}{e}_k$$

Eigenvectors can be chosen s.t. $\underset{\sim}{e}_1'\underset{\sim}{e}_1 = \cdots = \underset{\sim}{e}_k'\underset{\sim}{e}_k = 1$

$\underset{\sim}{e}_i'\underset{\sim}{e}_j = 0 \quad \forall\ i \neq j$ (orthogonal)

Eigenvectors are unique unless 2 or more eigenvalues are equal.

Example 2.9 (verify eigenvalues/eigenvectors)

$$A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{2}} \\ -\frac{6}{\sqrt{2}} \end{bmatrix} = 6 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 6, \quad \underset{\sim}{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \underset{\sim}{e}_1' \underset{\sim}{e}_1 = \frac{1}{2} + \frac{1}{2} = 1$$

Second eigenvector/eigenvalue pair:

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{4}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \end{bmatrix} = \underset{\lambda}{-4} \underset{\underset{\sim}{e}_2}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$\underset{\sim}{e}_2' \underset{\sim}{e}_2 = \frac{1}{2} + \frac{1}{2} = 1 \qquad \underset{\sim}{e}_1' \underset{\sim}{e}_2 = \frac{1}{2} - \frac{1}{2} = 0$$

$$|A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - (-5)(-5) = 0 \Rightarrow 1 - 2\lambda + \lambda^2 - 25 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 24 = 0 \Rightarrow \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-24)}}{2(1)}$$

$$\Rightarrow \lambda = \frac{\cancel{2} \pm \sqrt{4+96}}{2} = \frac{\cancel{2} \pm 10}{2} = \frac{12}{2} = 6 = \lambda_1$$

$$\frac{-8}{2} = -4 = \lambda_2$$

$$A\underset{\sim}{x}_1 = \lambda_1 \underset{\sim}{x}_1$$

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} x_1 - 5x_2 = 6x_1 \\ -5x_1 + x_2 = 6x_2 \end{matrix}$$

$$\Rightarrow \begin{matrix} -5x_2 = 5x_1 \\ -5x_1 = 5x_2 \end{matrix} \Rightarrow x_1 = -x_2$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ making $\underset{\sim}{e}'\underset{\sim}{e} = 1 \Rightarrow \underset{\sim}{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$A\underset{\sim}{x}_2 = \lambda_2 \underset{\sim}{x}_2$$

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} -4x_1 \\ -4x_2 \end{matrix}$$

$$\begin{matrix} x_1 - 5x_2 = -4x_1 \\ -5x_1 + x_2 = -4x_2 \end{matrix} \Rightarrow \begin{matrix} 5x_1 = 5x_2 \\ 5x_2 = 5x_1 \end{matrix} \Rightarrow x_1 = x_2$$

$$\underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underset{\sim}{e}_2'\underset{\sim}{e}_2 = 1 \Rightarrow \underset{\sim}{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

2.3 Positive Definite Matrices

Quadratic forms that are always nonnegative and the associated positive definite matrices.

Spectral decomposition of $k \times k$ Symmetric A

$$A = \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2' + \cdots + \lambda_k \underset{\sim}{e}_k \underset{\sim}{e}_k'$$

$\lambda_1, \ldots, \lambda_k$ = eigenvalues $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_k$ orthogonal normalized eigenvectors

Give a numeric example (not textbook)

Quadratic form: $\underset{\sim}{x}' A \underset{\sim}{x}$ (symmetric $k \times k$ A)

has only squared terms x_i^2 and product terms $x_i x_k$

$0 \leq \underset{\sim}{x}' A \underset{\sim}{x} \quad \forall \; \underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \Rightarrow \underset{\sim}{x}' A \underset{\sim}{x}$ and A are said to be nonnegative definite.

If the "=" holds only when $\underset{\sim}{x} = \underset{\sim}{0}$, it is positive definite. $\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} > 0$

Example 2.11 (Pos. Def matrix and Quadratic form)

$$x'Ax = 3x_1^2 + 2x_2^2 - 2\sqrt{2}\, x_1 x_2 = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2$$

$$A = \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$$

$$\underset{\sim}{x}'A\underset{\sim}{x} = (\underset{\sim}{x}'A)\underset{\sim}{x} = \begin{bmatrix} a_{11}x_1 + \frac{1}{2}a_{12}x_2 & \frac{1}{2}a_{12}x_1 + a_{22}x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= a_{11}x_1^2 + \frac{1}{2}a_{12}x_2x_1 + \frac{1}{2}a_{12}x_1x_2 + a_{22}x_2^2 = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$

$$A = \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2'$$

A : positive definite $\Leftrightarrow$ every eigenvalue of $A > 0$

A : non-negative " $\Leftrightarrow$ " " " " ≥ 0

$$(\text{distance})^2 = \sum_{i=1}^{p} a_{ii}x_i^2 + 2\sum\sum_{i<i'} a_{ii'}x_i x_{i'} > 0 \quad \forall\, \underset{\sim}{x} \neq 0$$

$a_{ii'} = a_{i'i}$

$\Rightarrow$ distance : a positive definite quadratic form

$x'Ax$

Squared distance to arbitrary fixed point $\underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix}$ is $(\underset{\sim}{x}-\underset{\sim}{\mu})' A (\underset{\sim}{x}-\underset{\sim}{\mu})$

Geometric Interpretation of eigenvalues/vectors

$p=2$: Constant distance $d^2 = \underset{\sim}{x}' A \underset{\sim}{x} = a_{11} x_1^2 + a_{22} x_2^2 + 2a_{12} x_1 x_2 = c^2$

$A = \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2'$

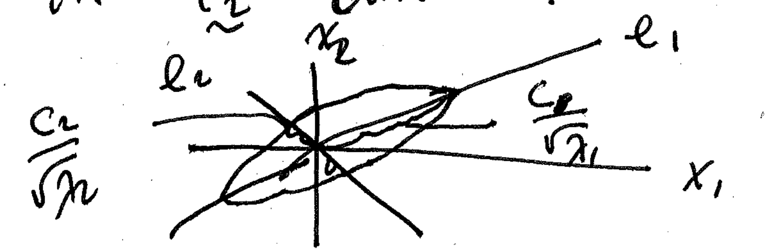
$\Rightarrow \underset{\sim}{x}' A \underset{\sim}{x} = \lambda_1 (\underset{\sim}{x}' \underset{\sim}{e}_1)^2 + \lambda_2 (\underset{\sim}{x}' \underset{\sim}{e}_2)^2$ (since $\underset{\sim}{e}_1' \underset{\sim}{e}_2 = 0$)

$\Rightarrow c^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2 \qquad y_1 = \underset{\sim}{x}_1' \underset{\sim}{e}_1 \quad y_2 = \underset{\sim}{x}_2' \underset{\sim}{e}_2$

$\lambda_1, \lambda_2 > 0$ when $A \equiv p.d.$

$\underset{\sim}{x} = c \lambda_1^{-1/2} \underset{\sim}{e}_1$ ~~satisf~~ satisfies $x' A x = \lambda_1 (c \lambda_1^{-1/2} \underset{\sim}{e}_1' \underset{\sim}{e}_1)^2 = c^2$

and $\underset{\sim}{x} = c \lambda_2^{-1/2} \underset{\sim}{e}_2$ gives appropriate distance in $\underset{\sim}{e}_2$ direction.



2.4 Square Root Matrix Based on Spectral Decomposition

$A \equiv k \times k$ pos. def. matrix w/ the Spectral Decomposition

$A = \sum_{i=1}^{k} \lambda_i \underset{\sim}{e}_i \underset{\sim}{e}_i'$ define $P = [\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_k]$

$A = \sum_{i=1}^{k} \lambda_i \underset{\sim}{e}_i \underset{\sim}{e}_i' = P \Lambda P'$ $\Lambda = \text{diag}\{\lambda_i\}$ $(\lambda_i > 0)$

Note: $P\Lambda = [e_1 \; e_2 \; \cdots \; e_p] \Lambda = [\underset{\sim}{e}_1 \lambda_1 \; \underset{\sim}{e}_2 \lambda_2 \cdots \underset{\sim}{e}_p \lambda_p]$

$= [\lambda_1 \underset{\sim}{e}_1 \; \lambda_2 \underset{\sim}{e}_2 \; \ldots \; \lambda_p \underset{\sim}{e}_p]$

$\Rightarrow P\Lambda P' = [\lambda_1 \underset{\sim}{e}_1 \; \lambda_2 \underset{\sim}{e}_2 \cdots \lambda_p \underset{\sim}{e}_p] \begin{bmatrix} \underset{\sim}{e}_1' \\ \underset{\sim}{e}_2' \\ \vdots \\ \underset{\sim}{e}_p' \end{bmatrix}$

$= \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2' + \cdots + \lambda_p \underset{\sim}{e}_p \underset{\sim}{e}_p'$

Note $P'P = \begin{bmatrix} e_1' \\ e_2' \\ \vdots \\ e_p' \end{bmatrix} [\underset{\sim}{e}_1 \; \underset{\sim}{e}_2 \; \ldots \; \underset{\sim}{e}_p] = \begin{bmatrix} \underset{\sim}{e}_1'\underset{\sim}{e}_1 & \underset{\sim}{e}_1'\underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_1'\underset{\sim}{e}_p \\ \underset{\sim}{e}_2'\underset{\sim}{e}_1 & \underset{\sim}{e}_2'\underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_2'\underset{\sim}{e}_p \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{e}_p'\underset{\sim}{e}_1 & \underset{\sim}{e}_p'\underset{\sim}{e}_2 & \cdots & \underset{\sim}{e}_p'\underset{\sim}{e}_p \end{bmatrix} = I$

$\cancel{e_{11}^2 + e_{21}^2 + \cdots + e_{p1}^2 \quad e_{11}e_{12} + e_{21}e_{22} + e}$

$PP' = [\underset{\sim}{e}_1 \; \underset{\sim}{e}_2 \cdots \underset{\sim}{e}_p] \begin{bmatrix} \underset{\sim}{e}_1' \\ \underset{\sim}{e}_2' \\ \vdots \\ e_p' \end{bmatrix} = I$ (showed on computer)

$A^{-1} = P\Lambda^{-1}P' = \sum_i \frac{1}{\lambda_i} \underset{\sim}{e}_i \underset{\sim}{e}_i'$

$(P\Lambda^{-1}P')(P\Lambda P') = (P\Lambda P')(P\Lambda^{-1}P') = P\Lambda\Lambda^{-1}P' = I$

$$A^{1/2} = \sum_i \sqrt{\lambda_i}\, \underset{\sim}{e}_i \underset{\sim}{e}_i' = P\Lambda^{1/2}P'$$

1) $(A^{1/2})' = A^{1/2}$ (symmetric)

2) $A^{1/2}A^{1/2} = A$

3) $(A^{1/2})^{-1} = \sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}} \underset{\sim}{e}_i \underset{\sim}{e}_i' = P\Lambda^{-1/2}P'$ where $\Lambda^{-1/2} = \mathrm{diag}\left\{\frac{1}{\sqrt{\lambda_i}}\right\}$

4) $A^{1/2}A^{-1/2} = A^{-1/2}A^{1/2} = I$, $A^{-1/2}A^{-1/2} = A^{-1}$ where $A^{-1/2} = (A^{1/2})^{-1}$

2.5 Random Vectors and Matrices

$$E\{X\} = \begin{bmatrix} E\{X_{11}\} & \cdots & E\{X_{1p}\} \\ \vdots & \ddots & \vdots \\ E\{X_{n1}\} & \cdots & E\{X_{np}\} \end{bmatrix}$$

X, Y conformable and random $\quad E\{X+Y\} = E\{X\} + E\{Y\}$

A, X, B conformable: $\quad E\{AXB\} = AE\{X\}B$
A, B constants

2.6 Mean Vectors and Covariance Matrices

vector $\underset{\sim}{X}$ $\quad E\{\underset{\sim}{X}\} = \begin{bmatrix} E\{X_1\} \\ \vdots \\ E\{X_p\} \end{bmatrix} = \mu$

$$\Sigma = E[(X-\mu)(X-\mu)'] = E\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ \vdots & & & \\ (X_p-\mu_p)(X_1-\mu_1) & \cdots & \cdots & (X_p-\mu_p)^2 \end{bmatrix}$$

~~Std. Deviation Matrix~~

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix} \qquad \rho = \left\{ \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \right\} \quad i,j=1,\dots,p$$

Standard Deviation Matrix $V^{1/2} = \operatorname{diag}\{\sigma_{ii}\}$

$$V^{1/2} \rho V^{1/2} = \Sigma \qquad \rho = V^{-1/2} \Sigma V^{-1/2} = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$$

Partitioning the Covariance Matrix

Set of p characteristics may be naturally separated into 2 or more subjects of the total set of p characteristics. Total collection $\equiv (p \times 1)$-dim random vector X, subsets are components of X, and sorted by partitioning X.

Suppose 2 groups of size q, $p-q$

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ \cdots \\ X^{(2)} \end{bmatrix} \qquad \mu = E\{X\} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \hline \mu^{(2)} \end{bmatrix}$$

$$(X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})' = \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ \vdots \\ X_q-\mu_q \end{bmatrix} [X_{q+1}-\mu_{q+1} \cdots X_p-\mu_p]$$

$$= \begin{bmatrix} (X_1-\mu_1)(X_{q+1}-\mu_{q+1}) & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ (X_2-\mu_2)(X_{q+1}-\mu_{q+1}) & & (X_2-\mu_2)(X_p-\mu_p) \\ \vdots & \ddots & \vdots \\ (X_q-\mu_q)(X_{q+1}-\mu_{q+1}) & & (X_q-\mu_q)(X_p-\mu_p) \end{bmatrix}$$

$$\Rightarrow E\{(X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})'\} = \begin{bmatrix} \sigma_{1,q+1} & \cdots & \sigma_{1p} \\ \sigma_{2,q+1} & \cdots & \sigma_{2p} \\ \vdots & \ddots & \vdots \\ \sigma_{q,q+1} & \cdots & \sigma_{qp} \end{bmatrix}$$

$$(X-\mu)(X-\mu)' = \begin{bmatrix} (X^{(1)}-\mu^{(1)}) \\ (X^{(2)}-\mu^{(2)}) \end{bmatrix} [(X^{(1)}-\mu^{(1)})' \quad (X^{(2)}-\mu^{(2)})']$$

$$= \begin{bmatrix} (X^{(1)}-\mu^{(1)})(X^{(1)}-\mu^{(1)})' & (X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})' \\ (X^{(2)}-\mu^{(2)})(X^{(1)}-\mu^{(1)})' & (X^{(2)}-\mu^{(2)})(X^{(2)}-\mu^{(2)})' \end{bmatrix}$$

$$\Rightarrow \Sigma = E\{(X-\mu)(X-\mu)'\} = \begin{array}{c c} & \begin{array}{cc} q & p-q \end{array} \\ \begin{array}{c} q \\ p-q \end{array} & \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \end{array}$$

where $\Sigma_{12} = \Sigma_{21}'$

Mean Vector and Covariace Matrix for Linear Combinations of Random Variables

$$E\{cX_1\} = cE\{X_1\} = c\mu_1$$

$$V\{cX_1\} = E\{(cX_1 - c\mu_1)^2\} = c^2 E\{(X_1-\mu_1)^2\} = c^2\sigma_{11}$$

$$Cov\{aX_1, bX_2\} = E\{(aX_1 - a\mu_1)(bX_2 - b\mu_2)\}$$

$$= ab\, E\{(X_1-\mu_1)(X_2-\mu_2)\} = ab\,\sigma_{12}$$

$$E\{aX_1 + bX_2\} = aE\{X_1\} + bE\{X_2\} = a\mu_1 + b\mu_2$$

$$V\{aX_1 + bX_2\} = E\{[(aX_1 + bX_2) - (a\mu_1 + b\mu_2)]^2\}$$

$$= E\{[a(X_1-\mu_1) + b(X_2-\mu_2)]^2\}$$

$$= E\{[a^2(X_1-\mu_1)^2 + b^2(X_2-\mu_2)^2 + 2ab(X_1-\mu_1)(X_2-\mu_2)]\}$$

$$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\,\sigma_1\sigma_2$$

$$c' = [a \;\; b] \Rightarrow c'X = [a \;\; b]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = aX_1 + bX_2$$

$$E\{aX_1 + bX_2\} = a\mu_1 + b\mu_2 = [a \;\; b]\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = c'\underset{\sim}{\mu}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \Rightarrow V\{aX_1 + bX_2\} = V\{c'X\} = \underset{\sim}{c}'\Sigma\underset{\sim}{c}$$

since $c'\Sigma = [a\sigma_{11} + b\sigma_{12} \quad a\sigma_{12} + b\sigma_{22}]$

$$\Rightarrow c'\Sigma c = c'\Sigma \begin{bmatrix} a \\ b \end{bmatrix} = a^2\sigma_{11} + ab\sigma_{12} + ba\sigma_{12} + b^2\sigma_{22}$$

$$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12}$$

$$\boxed{\begin{array}{l} c'X = c_1X_1 + c_2X_2 + \dots + c_pX_p \\ \text{Mean: } E\{c'X\} = c'\mu \\ \text{Variance: } V\{c'X\} = c'\Sigma c \end{array}}$$

q linear combinations of $X_1, \dots, X_p$

$$Z_1 = c_{11}X_1 + c_{12}X_2 + \dots + c_{1p}X_p$$

$$\vdots$$

$$Z_q = c_{q1}X_1 + c_{q2}X_2 + \dots + c_{qp}X_p$$

$$\equiv \quad \underset{\sim}{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_q \end{bmatrix} = \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_q' \end{bmatrix} X = CX$$

$$\mu_Z = E\{Z\} = E\{CX\} = C\,E\{X\}$$

$$\Sigma_Z = Cov\{Z\} = Cov\{CX\} = C\,\Sigma_X\, C'$$

Example 2.15

$$Z_1 = X_1 - X_2 \qquad Z_2 = X_1 + X_2$$

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = CX \quad C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mu_Z = E\{Z\} = C\mu_X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix}$$

$$\Sigma_Z = \mathrm{Cov}\{Z\} = C\Sigma_X C' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{12} & \sigma_{12} - \sigma_{22} \\ \sigma_{11} + \sigma_{12} & \sigma_{12} + \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} + 2\sigma_{12} + \sigma_{22} \end{bmatrix}$$

Partitioning the Sample Mean Vector and Covariance Matrix

$$\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix} \qquad S_n = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & & \\ s_{1p} & \cdots & s_{pp} \end{bmatrix}$$

Partitioned:

$$\bar{X} = \begin{array}{c} \begin{bmatrix} \bar{X}^{(1)} \\ \bar{X}^{(2)} \end{bmatrix} \begin{array}{l} q \\ p-q \end{array} \end{array} \qquad S_n = \begin{array}{cc} & \begin{array}{cc} q & p-q \end{array} \\ \begin{array}{c} q \\ p-q \end{array} & \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right] \end{array}$$

$$S_{21} = S_{12}'$$

2.7 Matrix Inequalities and Maximization

Cauchy-Schwarz Inequality: $\underset{\sim}{b}, \underset{\sim}{d}$ any $p \times 1$ vectors:

Then $(\underset{\sim}{b}'\underset{\sim}{d})^2 \leq (\underset{\sim}{b}'\underset{\sim}{b})(\underset{\sim}{d}'\underset{\sim}{d})$ w/ equality only if $\underset{\sim}{b} = c\underset{\sim}{d}$ for some constant c.

Extended Cauchy-Schwarz Inequality: $\underset{p\times 1}{\underset{\sim}{b}}, \underset{p\times 1}{\underset{\sim}{d}}$ $\underset{p\times p}{B} \equiv$ pos.-def.

$(b'd)' \leq (b'Bb)(d'B^{-1}d)$ w/ equality iff $\underset{\sim}{b} = cB^{-1}d$ or $\underset{\sim}{d} = cB\underset{\sim}{d}$ for some constant c.

Maximization Lemma $\underset{p\times p}{B} \equiv$ P.d. $\underset{p\times 1}{\underset{\sim}{d}} \equiv$ given vector

$\underset{p\times 1}{x} \equiv$ arbitrary non-zero vector

$$\max_{x \neq 0} \frac{(x'd)^2}{x'Bx} = \underset{\sim}{d}'B^{-1}\underset{\sim}{d} \quad @ \ \underset{\sim}{x} = cB^{-1}\underset{\sim}{d}$$

for any constant $c \neq 0$

Maximization of Quadratic Forms for Points on Unit Sphere

$\underset{p\times p}{B} \equiv$ P.d. matrix w/ eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$ and assoc. eigenvectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_p$ Then:

$$\max_{x \neq 0} \frac{x'Bx}{x'x} = \lambda_1 \quad (\text{attained @ } \underset{\sim}{x} = \underset{\sim}{e}_1)$$

$$\min_{x \neq 0} \frac{x'Bx}{x'x} = \lambda_p \quad (\text{" " } \underset{\sim}{x} = \underset{\sim}{e}_p)$$

Also $$\max_{x \perp \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_k} \frac{x'Bx}{x'x} = \lambda_{k+1} \quad (\text{attained @ } \underset{\sim}{e}_{k+1}) \quad k = 1, \ldots, p-1$$