Chapter 4

21. X : length of one side $\quad X \sim \text{Uniform}[0, 1]$

$$EX^2 = Var(X) + (EX)^2 = \frac{1}{12} + \left(\frac{1}{2}\right)^2 = \frac{1}{3}$$

or $\quad EX^2 = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

22. $X \sim \text{Uniform}[0, 1] \quad Y \sim \text{Uniform}[0, 1]$

X, Y are independent

Area of the rectangle is $X \cdot Y$

$$EX \cdot Y = EX \cdot EY = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

This area is less than the area in Problem 21

31. $X \sim \text{Uniform}[1, 2]$

$$E\frac{1}{X} = \int_1^2 \frac{1}{x} \cdot 1\, dx = \ln x\Big|_1^2 = \ln 2$$

$$EX = \frac{3}{2} = 1.5 \quad \left(EX = \int_1^2 x\, dx = \frac{x^2}{2}\Big|_1^2 = 1.5 \right)$$

$$\frac{1}{EX} = \frac{2}{3} \neq E\frac{1}{X}$$

48 (a) $Cov(U, V) = Cov(Z+X, Z+Y) = Cov(Z, Z) + Cov(Z, Y)$
$+ Cov(X, Z) + Cov(X, Y) = \sigma_Z^2 + 0 + 0 + 0 = \sigma_Z^2$

Because X, Y, Z are uncorrelated random variables.

(b) $\rho_{UV} = \dfrac{Cov(U, V)}{\sqrt{Var(U) \cdot Var(V)}}$

$$Var(U) = Var(Z+X) = Var\,Z + Var\,X = \sigma_Z^2 + \sigma_X^2$$

$$Var(V) = Var(Z+Y) = Var\,Z + Var\,Y = \sigma_Z^2 + \sigma_Y^2$$

$$\rho_{UV} = \frac{\sigma_Z^2}{\sqrt{(\sigma_Z^2 + \sigma_X^2) \cdot (\sigma_Z^2 + \sigma_Y^2)}}$$