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# ESTIMATORS WITH NONDECREASING RISK: APPLICATION OF A CHI-SQUARED IDENTITY

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*Abstract*: By using an apparently little known fact about concave functions together with a new expectation identity for noncentral chi-squared random variables, a characterization of risk functions of Stein-type estimators is obtained. In particular, concavity of the function appearing in the shrinkage factor is related to the estimator's risk function being nondecreasing.

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## 1. Introduction

Let $X$ be an observation from a $p$-variate ($p \geqslant 3$) normal distribution with mean $\theta$ and identity covariance matrix. For any estimator $\delta(X)$ of $\theta$, the loss in estimating $\theta$ by $\delta(X)$ is

$$L(\theta, \delta) = |\theta - \delta(X)|^2,$$

where $|\cdot|$ denotes Euclidean distance. The risk of $\delta(X)$, $R(\theta, \delta)$, is given by

$$R(\theta, \delta) = E_\theta L(\theta, \delta) = E_\theta|\theta - \delta(X)|^2,$$

and $\delta(X)$ is a minimax estimator of $\theta$ if and only if $R(\theta, \delta) \leqslant R(\theta, X) = p$ for all $\theta$. Many Stein-type estimators have the form

$$\delta(X) = \left(1 - r(|X|^2)/|X|^2\right)X,$$

and theorems about the risk behavior of $\delta(X)$ give conditions on $r(|X|)$. In particular, common conditions are that $r(t)$ be nondecreasing and $r(t)/t$ be nonincreasing. The following lemma relates the property of concavity to these conditions.

**Lemma 1.** *Let $r: [0, \infty) \to [0, \infty)$ be concave. Then*
(i) *$r(t)$ is nondecreasing,*
(ii) *$r(t)/t$ is nonincreasing.*

**Proof.** Since $r(\cdot)$ is concave, it follows that

$$r((1-\lambda)t_1 + \lambda t_2) \geqslant (1-\lambda)r(t_1) + \lambda r(t_2)$$

for $0 \leqslant \lambda \leqslant 1$. Moreover, this inequality is reversed if $\lambda > 1$.

We now prove part (i) by contradiction. Suppose $t_1 < t_2$ and $r(t_1) > r(t_2)$. Then the function of $\lambda$, $f(\lambda) = (1-\lambda)r(t_1) + \lambda r(t_2)$ is decreasing and, for sufficiently large $\lambda > 1$, it is negative. For sufficiently large $\lambda$ we then have

$$r((1-\lambda)t_1 + \lambda t_2) \leqslant (1-\lambda)r(t_1) + \lambda r(t_2) < 0,$$

which contradicts the fact that $r(\cdot)$ is nonnegative on $[0, \infty)$ and establishes part (i).

To prove part (ii) write, for $0 < t_1 < t_2$,

$$t_1 = \left[1 - \frac{t_1}{t_2}\right] \times 0 + \left[\frac{t_1}{t_2}\right] \times t_2.$$

By the concavity of $r(\cdot)$ we have

$$r(t_1) = r\left(\left[1 - \frac{t_1}{t_2}\right] \times 0 + \left[\frac{t_1}{t_2}\right] \times t_2\right)$$
$$\geqslant \left[1 - \frac{t_1}{t_2}\right] r(0) + \left[\frac{t_1}{t_2}\right] r(t_2) \geqslant \left[\frac{t_1}{t_2}\right] r(t_2),$$

where the second inequality follows from the fact that $r(0) \geqslant 0$. Thus $r(t_1)/t_1 \geqslant r(t_2)/t_2$ for $t_1 < t_2$, establishing part (ii). $\square$

Therefore, the two original minimax conditions are tied together through the property of concavity. The converse of Lemma 1 is false, so there can exist minimax estimators satisfying (i) and (ii) with nonconcave $r(t)$. However, most familiar estimators do, in fact, have concave $r(t)$.

Although the characterization in Lemma 1 may seem new, similar results are well known. For example, Barlow and Proschan (1975) define a function $g(t)$ on $[0, \infty)$ to be *starshaped* if $g(t)/t$ is increasing. They then give an exercise to show that convex functions on $[0, \infty)$ are star-shaped, which is quite close to the result of Lemma 1.

## 2. A chi-squared expectation identity

In the following, let $\chi_p^2$ denote a noncentral chi-squared random variable with $p$ degrees of freedom and noncentrality parameter $\frac{1}{2}|\theta|^2$. The value of $\theta$ is indicated by the subscript on the expectation operator.

**Lemma 2.** *Let $h:[0, \infty) \to (-\infty, \infty)$ be differentiable. Then, provided both sides exist,*

$$\frac{\partial}{\partial |\theta|^2} E_\theta\left[h\left(\chi_p^2\right)\right] = E_\theta\left[\frac{\partial}{\partial \chi_{p+2}^2} h\left(\chi_{p+2}^2\right)\right].$$

**Proof.** The lemma is established by equating the results of the well-known integration-by-parts technique with the results of some lesser-known identities for expectations of noncentral chi-squared random variables. We will proceed by evaluating the risk of the estimator $\delta(X) = (1 - [h(|X|^2)/|X|^2])X$, where $X \sim \mathrm{N}_p(\theta, I)$. The usual integration-by-parts yields

$$\begin{aligned} E_\theta&|\theta - \delta(X)|^2 \\ &= p - 4E_\theta h'(|X|^2) \\ &\quad + E_\theta\left\{\frac{h(|X|^2)}{|X|^2}\left[h(|X|^2) - 2(p-2)\right]\right\}. \end{aligned} \tag{1}$$

We can also write

$$\begin{aligned} E_\theta&|\theta - \delta(X)|^2 \\ &= E_\theta|(\theta - X) + (X - \delta(X))|^2 \\ &= p + 2E_\theta\{(\theta - X)' Xh(|X|^2)/|X|^2\} \\ &\quad + E_\theta\{h^2(|X|^2)/|X|^2\}. \end{aligned} \tag{2}$$

We now employ the following identities, which can be found either in Bock (1975) or Casella (1980). If $h:[0, \infty) \to (-\infty, \infty)$, then provided the expectations exist,

$$E_\theta\{Xh(|X|^2)\} = \theta E_\theta\left\{h\left(\chi_{p+2}^2\right)\right\}, \tag{3}$$

$$\begin{aligned} |\theta|^2 & E_\theta\left\{h\left(\chi_{p+2}^2\right)/\chi_{p+2}^2\right\} \\ &= E_\theta\left\{h\left(\chi_{p-2}^2\right)\right\} - (p-2)E_\theta\left\{h\left(\chi_p^2\right)/\chi_p^2\right\}, \end{aligned} \tag{4}$$

$$\begin{aligned} &\frac{\partial}{\partial |\theta|^2} E_\theta\left\{h\left(\chi_p^2\right)\right\} \\ &= \tfrac{1}{2}\left\{E_\theta h\left(\chi_{p+2}^2\right) - E_\theta h\left(\chi_p^2\right)\right\}. \end{aligned} \tag{5}$$

Now, using (3) and (4) on the first expectation in (2), and rearranging terms, we obtain

$$\begin{aligned} E_\theta&|\theta - \delta(X)|^2 \\ &= p - 2\left\{E_\theta h\left(\chi_p^2\right) - E_\theta h\left(\chi_{p-2}^2\right)\right\} \\ &\quad + E_\theta\left\{\frac{h(|X|^2)}{|X|^2}\left[h(|X|^2) - 2(p-2)\right]\right\}. \end{aligned} \tag{6}$$

We note in passing that, using the fact that the noncentral chi-squared distribution has monotone likelihood ratio in its degrees of freedom, equation (6) provides an immediate proof of the minimaxity of $\delta(X)$ provided $h(\cdot)$ is nondecreasing and $0 \leqslant h \leqslant 2(p-2)$.