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A proportional odds model with subject-specific effects for repeated ordered categorical responses

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SUMMARY

Suppose subjects make repeated responses on the same ordered categorical scale. We propose a generalization of the Rasch model that expresses the cumulative logit of the response distribution using subject parameters and a proportional odds structure for item effects. Parameters in the model describe subject-specific, rather than population-averaged, effects. Consistent estimation of the effects requires eliminating the subject parameters. We accomplish this by simultaneous fitting of Rasch models, conditional on sufficient statistics for those parameters, for the possible binary collapsings of the response. The fitting process uses an improved Newton–Raphson algorithm for fitting generalized loglinear models by maximum likelihood estimation subject to constraints. For the case of two items, we give simple expressions for an effect estimate and its standard error, and suggest a test of marginal homogeneity for ordinal matched pairs.

Some key words: Conditional estimation; Constrained maximum likelihood; Cumulative logit model; Item response model; Marginal homogeneity; Matched pairs; Ordinal data; Quasi symmetry; Random effects; Rasch model; Square contingency table.

1. INTRODUCTION

Suppose N subjects make k repeated responses on the same ordered categorical scale. For instance, for k similar questions in a survey, subjects might make responses on a scale such as (strongly agree, mildly agree, mildly disagree, strongly disagree). For subject $i\,(i = 1, \ldots, N)$ and response measurement $j\,(j = 1, \ldots, k)$, let ϕ_{hij} denote the probability of response in category h, for $h = 1, \ldots, r$. For $h = 1, \ldots, r-1$, let $\gamma_{hij} = \phi_{1ij} + \ldots + \phi_{hij}$, and consider the cumulative logit model

$$\log\{\gamma_{hij}/(1-\gamma_{hij})\} = \lambda_h + \alpha_i - \beta_j. \tag{1·1}$$

Identifiability requires a constraint on two of the sets of parameters, such as $\lambda_1 = \beta_1 = 0$.

Model (1·1) is a proportional odds model, the subject and response effects being independent of h. The model holds if, for each i and j, there is an underlying continuous response that has a logistic distribution with mean $\beta_j - \alpha_i$, and the observed response falls in category h when the underlying response falls between λ_{h-1} and λ_h. Normally the 'cutpoints' $\{\lambda_h\}$ and the $\{\alpha_i\}$ are incidental parameters, and estimation of $\{\beta_j\}$ is paramount. In many applications, different subjects may use different cutpoints. For instance, for a given value for the underlying response, one subject may regard it as falling into category 'good', while a second subject regards it as 'very good'. Because of this and because the response effects $\{\beta_j\}$ are of primary interest, we mainly consider the more general model

$$\log\{\gamma_{hij}/(1-\gamma_{hij})\} = \alpha_{hi} - \beta_j, \tag{1·2}$$

where we use the constraint $\beta_1 = 0$.

For each subject, the distributions for the k responses in these models are stochastically ordered according to $\{\beta_j\}$. For $r = 2$, models (1·1) and (1·2) simplify to the Rasch item-response model (Rasch, 1961). To emphasize connections with that model, we refer to the k response characteristics as items, and we refer to model (1·2) as the cumulative Rasch model. McCullagh (1977) and Ezzet & Whitehead (1991) discussed the ordinal-response case for $k = 2$, and Samejima (1969), Andersen (1980), Masters (1982), Tutz (1990) and Agresti (1993) discussed alternative ordinal item-response models.

For a given subject i with fixed α_i or $\{\alpha_{hi}\}$, we assume that responses on separate items are independent. Subject heterogeneity implies that the joint distribution of the responses, averaged over subjects, shows positive associations between pairs of items. We also assume that responses by different subjects are independent. For the Rasch model, the unconditional maximum likelihood estimators of $\{\beta_j\}$ are inconsistent as $N \to \infty$ because of the concomitant increase in the number of subject parameters (Andersen, 1980, p. 244). One obtains consistency by calculating conditional maximum likelihood estimates of $\{\beta_j\}$, given sufficient statistics for $\{\alpha_i\}$. Tjur (1982) noted that one can obtain the conditional estimates by fitting a log linear model to the 2^k cross-classification of subjects' responses on the k items. Tjur's model is simply the quasi-symmetry model, and the conditional maximum likelihood estimates relate to ordinary maximum likelihood estimates of main effect parameters for that model.

Models (1·1) and (1·2) do not have reduced sufficient statistics, so the standard conditional approach is unavailable. For $k = 2$, McCullagh (1977) suggested an ingenious approach for obtaining consistency, using two weighted-average estimators of $\beta_2 - \beta_1$. The purpose of our note is to show a way to obtain consistent estimators of $\{\beta_j\}$ for arbitrary fixed k and r. We eliminate subject parameters by fitting the Rasch model using conditional maximum likelihood simultaneously for all binary collapsings of the response. This process corresponds to fitting a generalized quasi-symmetry model. We present a Newton–Raphson algorithm for fitting the generalized model, using maximum likelihood subject to constraints. For $k = 2$, a naive approach yields an appealling closed-form expression for an effect estimate and provides a simple way to test marginal homogeneity for ordinal matched pairs.

2. A CORRESPONDING GENERALIZED QUASI-SYMMETRY MODEL

For fixed h, the cumulative Rasch model (1·2) is the ordinary Rasch model for a collapsing of the response into $(\leqslant h, > h)$. Thus, when model (1·2) holds, one can consistently estimate $\{\beta_j\}$ using conditional maximum likelihood estimators for the Rasch model applied to the collapsed scale, for any h. Such estimates are inefficient, more so as r increases. A referee has pointed out that for $k = 2$ and an item effect near zero, an optimum selection of cutpoints for r categories can provide relative efficiency on the order of $1 - 1/r^2$ compared with $r = \infty$. Thus, for a large number of categories, the estimates from a binary collapsing could be about 75% as efficient as ones based on the full scale. We will present a more efficient approach for which estimates utilize the full scale. This approach also has the robustness benefit of providing protection against a lop-sided distribution of response probabilities in a single binary collapsing.

Consider a $(r-1) \times 2^k$ contingency table, in which the hth 2^k table cross-classifies responses for the hth binary collapsing of the scale, $h = 1, \ldots, r-1$. Each subject occurs in each 2^k table, so this table contains $N(r-1)$ observations. We refer to this table that simultaneously displays the $r-1$ collapsings as the collapsed table. For collapsing h, let $n(h; h_1, \ldots, h_k)$ denote the number of subjects making collapsed response h_j to item j, where $1 \leqslant h \leqslant r-1$, $1 \leqslant h_j \leqslant 2$, for $j = 1, \ldots, k$. Let $m(h; h_1, \ldots, h_k) = En(h; h_1, \ldots, h_k)$. We will estimate $\{\beta_j\}$ simultaneously using all collapsings, by fitting the model

$$\log m(h; h_1, \ldots, h_k) = \mu - \sum_j \beta_j I(h_j = 1) + \delta(h; h_1, \ldots, h_k) \qquad (2{\cdot}1)$$

to the collapsed table. Here, I denotes the indicator function and, for each h, $\delta(h; h_1, \ldots, h_k)$ is permutation invariant in $(h_1, \ldots, h_k)$. For fixed h, model (2·1) is the ordinary quasi-symmetry model for a 2^k table. The generalized quasi-symmetry model (2·1) is derived using a direct extension of the arguments given by Tjur (1982) relating the binary Rasch model to the quasi-symmetry model. The extension is straightforward, so we simply outline the main ideas.

For fixed h in the cumulative Rasch model, consider the probability of obtaining a particular sequence of collapsed responses. Suppose one eliminates the subject parameter by integrating this probability with respect to an unspecified distribution $F_h(\alpha)$. This leads to the likelihood used in a nonparametric marginal maximum likelihood solution for the binary model. Tjur noted that an extended version of this likelihood is equivalent to the likelihood for the quasi-symmetry model, apart from a term that occurs from assuming Poisson rather than multinomial sampling. Using Tjur's argument for each h and comparing the quasi-symmetry models for the various values of h, one observes structure (2·1).

For each fixed h, Tjur noted that the Poisson likelihood for model (2·1) can be decomposed as the product of a function of $\{\beta_j\}$ and a function of the other parameters, such that the function of $\{\beta_j\}$ is the conditional likelihood function for the Rasch model for that collapsed response. Thus, maximum likelihood estimates of $\{\beta_j\}$ and the second derivatives of the log likelihood with respect to them are identical, for that fixed h, for (2·1) and the conditional Rasch model. The issue of estimating $\{\beta_j\}$ in the cumulative Rasch model is not straightforward, however, when one considers the generalized model (2·1) simultaneously for all h; that is, for the full, rather than collapsed, response scale. Suppose we assume a multinomial or independent Poisson sampling model for cell counts in the original r^k cross classification of responses for the k items. Counts $\{n(h; h_1, \ldots, h_k)\}$ from different 2^k sections of the collapsed table do not follow such a sampling scheme, since the same subjects occur in each section.

To obtain consistent estimators of $\{\beta_j\}$ and of standard errors, one can maximize the Poisson or multinomial likelihood for the cell counts in the original r^k table subject to the constraint that model (2·1) holds for the collapsed $(r-1) \times 2^k$ table. Denote the cell counts in the original r^k table by the column vector n, and their expected frequencies by μ. Model (2·1) has generalized log linear form

$$\log A\mu = X\beta, \quad \text{ident}\,(\mu) = 0, \qquad (2{\cdot}2)$$

where $\text{ident}\,(\mu) = 0$ denotes a multinomial identifiability constraint $1'(\mu - n) = 0$ used for multinomial sampling. The $(r-1)2^k \times r^k$ matrix A, when applied to μ forms

$\{m(h; h_1, \ldots, h_k)\}$. To fit models of this form, it is very awkward to re-parameterize the cell probabilities in the multinomial likelihood in terms of the model parameters. An alternative and simpler method, particularly for large k, utilizes Lagrange's method of undetermined multipliers. We next present an improved algorithm for this method, developed in an unpublished Ph.D. dissertation at the University of Florida by J. Lang.

3. AN ALGORITHM FOR FITTING GENERALIZED LOG LINEAR MODELS

In Lagrange's method of undetermined multipliers, one views the model (2·2) as inducing constraints on cell probabilities, and one maximizes the kernel of the multinomial log likelihood, $l(\mu; n) = n' \log \mu$, subject to those constraints. The algorithm we present is a modification of one given by Haber (1985). Our algorithm uses an iterative scheme involving matrices that are simpler to invert, and it uses a re-parameterization from μ to $\xi = \log \mu$ to avoid interim out-of-range values during the iterative process.

Let U denote a full column rank matrix such that the space spanned by its columns is the orthogonal complement of the space spanned by the columns of X. Haber outlined a method for computing U. For $l(\xi; n) = n'\xi$, we express the parameter space for (2·2) as

$$\{\xi : U' \log A e^{\xi} = 0, \text{ident}\,(\xi) = 0\} = \{\xi : h(\xi) = 0, \text{ident}\,(\xi) = 0\}.$$

Let $\theta = \text{vec}\,(\xi, \lambda)$, where λ is a vector of undetermined multipliers that has as many elements as the rank of U. We solve for $\hat{\xi}$ by solving for $\hat{\theta}$ in the likelihood equations

$$g(\hat{\theta}) = \begin{pmatrix} \partial l(\hat{\xi}; n)/\partial \xi - e^{\hat{\xi}} + \{\partial h(\hat{\xi})'/\partial \xi\}\hat{\lambda} \\ h(\hat{\xi}) \end{pmatrix} = \begin{pmatrix} n - e^{\hat{\xi}} + H(\hat{\xi})\hat{\lambda} \\ h(\hat{\xi}) \end{pmatrix} = 0, \tag{3·1}$$

where

$$H(\xi) = \partial h(\xi)'/\partial \xi = \text{diag}\,(e^{\xi}) A' \,\text{diag}\,(A e^{\xi})^{-1} U$$

with diag (z) denoting a diagonal matrix with the elements of z on the main diagonal. Lang showed that the dominant part of $\partial g(\theta)/\partial \theta'$ is

$$G(\theta) = \begin{pmatrix} -\text{diag}\,(e^{\xi}) & H(\xi) \\ H(\xi)' & 0 \end{pmatrix}.$$

Specifically,

$$N^{-1} \frac{\partial g(\theta)}{\partial \theta'} = N^{-1} G(\theta) + \begin{pmatrix} o(1) & 0 \\ 0 & 0 \end{pmatrix},$$

where $o(1)$ represents a sequence that converges to zero as $N \rightarrow \infty$.

To solve the likelihood equations, we used the modified Newton–Raphson iterative scheme

$$\theta^{(t+1)} = \theta^{(t)} - \{G(\theta^{(t)})\}^{-1} g(\theta^{(t)}) \quad (t = 0, 1, 2, \ldots), \tag{3·2}$$

with $\xi^{(0)} = \log\,(n + \varepsilon)$ for some small $\varepsilon > 0$ and $\lambda^{(0)} = 0$. The advantage of using $G(\theta)$ is the simplicity of its inverse, which is

$$G^{-1}(\theta) = \begin{pmatrix} -D^{-1} + D^{-1}H(H'D^{-1}H)^{-1}H'D^{-1} & D^{-1}H(H'D^{-1}H)^{-1} \\ (H'D^{-1}H)^{-1}H'D^{-1} & (H'D^{-1}H)^{-1} \end{pmatrix},$$

where $D = \operatorname{diag}(e^{\xi})$ and $H = H(\xi)$. Using the delta method, we derived the asymptotic covariance matrices of the fitted values $\hat{\mu} = e^{\hat{\xi}}$ and of $\hat{\beta} = (X'X)^{-1}X' \log A\hat{\mu}$. Under multinomial sampling,

$$\Sigma_{\hat{\mu}} = D - (\mu\mu')/N - H(H'D^{-1}H)^{-1}H',$$

$$\Sigma_{\hat{\beta}} = (X'X)^{-1}X' \operatorname{diag}(A\mu)^{-1}A\Sigma_{\hat{\mu}}A' \operatorname{diag}(A\mu)^{-1}X(X'X)^{-1}.$$

One can check the quality of fit for a generalized log linear model using an ordinary goodness-of-fit test that compares observed and fitted counts in the original table. For the generalized quasi-symmetry model (2·1), the residual degrees of freedom equal $(r-1)2^k - r(k+1) + 2$. Lack of fit suggests that the cumulative Rasch model may be inappropriate. To analyze local lack of fit, one can define an adjusted residual for cell i by

$$e_i = (n_i - \hat{\mu}_i)/\textsc{ase}(n_i - \hat{\mu}_i),$$

where ASE denotes estimated asymptotic standard error. By the delta method, the estimated asymptotic covariance matrix of $(n - \hat{\mu})$ can be shown to equal

$$H(\hat{\xi})\{H(\hat{\xi})' \operatorname{diag}(\hat{\mu})^{-1}H(\hat{\xi})\}^{-1}H(\hat{\xi})'.$$

This is available as a by-product of the iterative scheme (3·2), and yields the estimated asymptotic standard errors needed for the adjusted residuals.

4. EXAMPLE

Table 1 is taken from the 1989 General Social Survey, conducted by the National Opinion Research Center at the University of Chicago. Subjects in the sample were asked their opinion on (1) early teens, age 14–16, having sex relations before marriage, (2) a man and a woman having sex relations before marriage, (3) a married person having sexual relations with someone other than the marriage partner. The response categories were 'always wrong', 'almost always wrong', 'wrong only sometimes', 'not wrong at all'.

For these data, the generalized quasi-symmetry model (2·1) corresponding to the cumulative Rasch model has likelihood-ratio goodness-of-fit statistic equal to 13·6 and Pearson statistic equal to 10·8, based on 10 degrees of freedom. Table 1 contains many zeros and small counts, so these summary statistics are crude indices of lack of fit. Fitted values are also shown in Table 1. The adjusted residuals showed no systematic pattern of lack of fit, and only four of them exceeded 2·0, with none exceeding 3·0. The estimates of $\{\beta_j\}$, scaled so that $\beta_1 = 0$, are $\hat{\beta}_2 = 4{\cdot}353$ (ASE = 0·339) and $\hat{\beta}_3 = -0{\cdot}548$ (ASE = 0·194), for which $\hat{\beta}_2 - \hat{\beta}_3 = 4{\cdot}901$ (ASE = 0·347). Responses regarding teen sex and extra-marital sex tended to be much more conservative than those regarding pre-marital adult sex.

5. A SIMPLE MEASURE OF MARGINAL HETEROGENEITY FOR ORDINAL MATCHED PAIRS

In the case of $k = 2$ items, a simple estimate of $\beta = \beta_2 - \beta_1$ results from the naive approach of treating different strata of the collapsed table as independent samples. The estimate obtained by fitting the quasi-symmetry model with homogeneous main effect terms to the $(r-1) \times (2 \times 2)$ collapsed table is identical to the estimate obtained for the 2×2 further collapsing of this table, collapsed over the cutpoint dimension. But this estimate is simply the log of the ratio of the two discordant counts in that table. In

Table 1. *Opinions about teenage sex, premarital sex, and extramarital sex, with fitted values for proportional odds Rasch model in parentheses*

Teen	Premarital	Extramarital sex			
sex	sex	1	2	3	4
1	1	140 (140)	1 (1·4)	0 (0·0)	0 (0·0)
	2	30 (30·4)	3 (3·1)	1 (0·8)	0 (0·0)
	3	66 (66·1)	4 (4·1)	2 (1·6)	0 (0·0)
	4	83 (83·1)	15 (15·5)	10 (7·9)	1 (3·8)*
2	1	3 (1·4)	1 (0·5)	0 (0·0)	0 (0·0)
	2	3 (3·0)	1 (1)	1 (0·8)	0 (0·0)
	3	15 (14·6)	8 (7·9)	0 (0·0)	0 (0·0)
	4	23 (22·4)	8 (7·9)	7 (5·4)	0 (0·0)
3	1	1 (0·9)	0 (0·0)	0 (0·0)	0 (0·0)
	2	0 (0·0)	0 (0·8)	0 (0·0)	0 (0·2)
	3	3 (3·5)	2 (2·4)	3 (3)	1 (1·1)
	4	13 (15·3)	4 (4·8)	6 (6·0)	0 (0·5)*
4	1	0 (0·0)	0 (0·0)	0 (0·0)	0 (0·0)
	2	0 (0·0)	0 (1·1)	0 (0·0)	0 (0·1)
	3	0 (0·0)	0 (0·0)	1 (1·0)	0 (0·0)
	4	7 (4·9)*	2 (1·4)	2 (1·3)*	4 (4)

* Adjusted residual exceeds 2·0.
Data from 1989 General Social Survey, with categories 1 = always wrong, 2 = almost always wrong, 3 = wrong only sometimes, 4 = not wrong.

terms of the cell counts $\{n_{ij}\}$ of the original $r \times r$ table, this estimate has the appealling form

$$\tilde{\beta} = \log\left[\left\{\sum_{i<j}(j-i)n_{ij}\right\}\Big/\left\{\sum_{i>j}(i-j)n_{ij}\right\}\right]. \tag{5·1}$$

Assuming the model holds, this is also a consistent estimator of β.

This estimator is related to one of McCullagh (1977), i.e. his Δ^*, with identical weights w_j^*. A standard error for $\tilde{\beta}$ that assumes independent strata is inappropriate, but it is simple to derive a proper one. Treating $\{n_{ij}\}$ in the original $r \times r$ table as a multinomial sample, the estimated asymptotic variance is

$$\hat{V}(\tilde{\beta}) = \sum_{i<j}(j-i)^2 n_{ij}\Big/\left\{\sum_{i<j}(j-i)n_{ij}\right\}^2 + \sum_{i>j}(i-j)^2 n_{ij}\Big/\left\{\sum_{i>j}(i-j)n_{ij}\right\}^2. \tag{5·2}$$

The ratio $\tilde{\beta}/\{\hat{V}(\tilde{\beta})\}^{1/2}$ is an approximately normally distributed statistic for testing that $\beta = 0$. This is a simple way to test marginal homogeneity in square ordinal tables when one expects the cumulative Rasch model to be approximately true. More generally, it is a reasonable statistic for ordinal matched-pairs data whenever we expect one response to be approximately a location shift of the other one. In our experience, the naive estimator (5·1) is quite adequate for a quick and simple analysis. It has little efficiency loss compared to the maximum likelihood one described in §4, in the sense that its estimated standard error is usually only slightly larger.

In the matched-pairs case, the test of fit we suggested for the log linear model corresponding to the cumulative Rasch model is simply a test that the $r-1$ collapsed tables of true probabilities all have the same ratio of discordant probabilities. In practice, one would conduct this $(r-2)$ degree-of-freedom test or at least examine the $r-1$ separate sample ratios before pooling them to form a summary index such as (5·1).

6. ALTERNATIVE FITTING APPROACHES AND ALTERNATIVE MODELS

An alternative approach to fitting cumulative Rasch models is parametric marginal maximum likelihood. One treats the subject parameters as random effects, and assumes some distribution for them. One integrates the likelihood with respect to that distribution to obtain a marginal likelihood free of the subject nuisance parameter. Ezzet & Whitehead (1991) used numerical integration to obtain marginal maximum likelihood fits for a random effects model of form (1·1) for $k=2$, assuming normally distributed subject effects. Harville & Mee (1984) and Jansen (1990) discussed related models.

We noted in § 2 that the generalized quasi-symmetry model (2·1) has structure motivated by an extension of a nonparametric marginal likelihood approach. One can obtain a more detailed marginal structure by using that approach with all possible combinations of cutpoints for the binary collapsings. Let $m(c_1,\ldots,c_k;\, h_1,\ldots,h_k)$ denote the expected number of subjects making collapsed response h_j to item j, when the cutpoint follows category c_j $(j=1,\ldots,k)$. The log linear model resulting from using this approach with the cumulative Rasch model (1·2) is a generalized quasi-symmetry model

$$\log m(c_1,\ldots,c_k;\, h_1,\ldots,h_k)=\mu-\sum_j \beta_j I(h_j=1)+\delta(c_1,\ldots,c_k;\, h_1,\ldots,h_k), \qquad (6{\cdot}1)$$

where the final term is permutation invariant for like permutations of $(h_1,\ldots,h_k)$ and $(c_1,\ldots,c_k)$.

When $k=2$, model (6·1) has a simple logit representation that also follows directly from model form (1·2). Let (Y_{i1}, Y_{i2}) denote the responses for subject i, and let

$$L_{ab}=\log\left\{\frac{P(Y_{i1}>a,\, Y_{i2}\leqslant b)}{P(Y_{i1}\leqslant a,\, Y_{i2}>b)}\right\}.$$

By independence of repeated responses for a given subject, the cumulative Rasch model (1·2) satisfies

$$L_{ab}=\operatorname{logit}(\gamma_{bi2})-\operatorname{logit}(\gamma_{ai1})=(\alpha_{bi}-\alpha_{ai})-(\beta_2-\beta_1),$$

so that

$$L_{ab}+L_{ba}=2(\beta_1-\beta_2), \qquad (6{\cdot}2)$$

for all $a\leqslant b$. The same relationship holds for the r^2 joint distributions averaged over subjects. In fact, when $k=2$, equation (6·2) characterizes the joint distribution corresponding to model (1·2). One can estimate the item effect by maximizing the multinomial likelihood for the r^2 observed table subject to constraint (6·2) holding for all $r(r-1)/2$ combinations of a and b. The special case with identical item effects, that is, $L_{ab}+L_{ba}=0$ for all $a\leqslant b$, is the complete symmetry model.

An alternative way to formulate subject-specific models for ordinal responses uses a model structure for which sufficient statistics exist for subject parameters. Agresti (1993) used the adjacent-categories logit in place of the cumulative logit in models (1·1) and (1·2), and obtained conditional maximum likelihood estimates by fitting a corresponding log linear model containing diagonal parameters.

A proportional odds model of somewhat different form from the one discussed in this paper utilizes 'population-averaged' effects for marginal distributions (Agresti, 1989).

The model is

$$\operatorname{logit}(\gamma_{h.j}) = \lambda_h - \beta_j, \tag{6·3}$$

where $\gamma_{h.j}$ denotes the probability of response $\leqslant h$ on item j for a randomly selected subject, that is, γ_{hij} averaged over the population of interest. This model refers to item distributions for an overall population, whereas the cumulative Rasch model applies to any set of subjects for whom item effects are identical. For Table 1, the estimates for model (6·3) are (0, 2·103, −0·335), with estimated standard errors of (0, 0·114, 0·104). We obtained these results by maximizing the multinomial likelihood for Table 1, subject to the constraint (6·3). These estimates are considerably different from the subject-specific estimates. As in the binary case, e.g. Neuhaus, Kalbfleisch & Hauck (1991), subject heterogeneity causes subject-specific effects to exceed population-averaged effects.

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